Computing the strength of some combinatorial theorems

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Notation

Definition

Given $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, $[X]^n = \{\text{size } n \text{ subsets of } X\}$.

Example

If $X = \{1, 2, 3\}$, $[X]^1 = \{\{1\}, \{2\}, \{3\}\}$, and
$[X]^2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Colorings and homogeneous sets

If $k \in \mathbb{N}$, we call $f : [X]^n \to \{1, \ldots, k\}$ a k-coloring of exponent n.

Definition

A subset $H \subseteq X$ is *homogeneous* if f is constant on $[H]^n$.

Ramsey's theorem

Fix $n, k \in \mathbb{N}$.

Theorem (Finite Ramsey's theorem)

For any $m \in \mathbb{N}$, there is a $w \in \mathbb{N}$ s.t.
for any $f : [\{1, \ldots, w\}]^n \to \{1, \ldots, k\}$,
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Theorem (Infinite Ramsey's theorem (RT^n_k))

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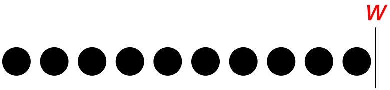

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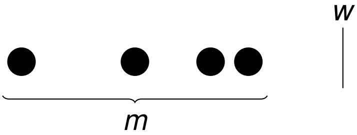

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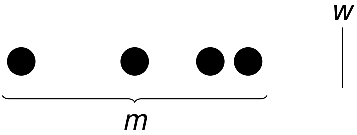

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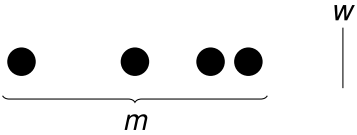

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Packed sets

Definition

Fix any $\phi : \mathbb{N} \to \mathbb{N}$, and any $A \subseteq \mathbb{N}$. *A is packed for* ϕ if

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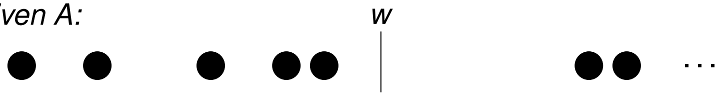

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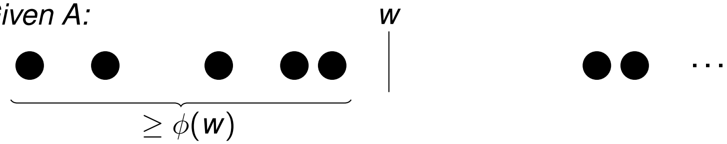

Packed Ramsey's theorem

Theorem (Erdős and Galvin)

Fix n and k, and any ϕ which grows sufficiently slowly[1]. For any $f : [\mathbb{N}]^n \to \{1, \ldots, k\}$, there is a set A which is

- *packed for ϕ, and*
- *given at most 2^{n-1} colors by f.[2]*

For fixed n and k as above, we refer to this theorem as PRT^n_k.

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[1] $w \to (\phi(w))^n_{k+1}$ for each w

[2] This is the best possible result: if ϕ is unbounded, there is a 2^{n-1} coloring that assigns all colors to any set packed for ϕ.

Comparing PRT and RT

Theorem (Flood)

Over RCA_0*,*

- PRT^1_k *is equivalent to* RT^1_k*,*
- PRT^n_k *is equivalent to* RT^n_k *for* $n \geq 3$ *and large enough* k*,*
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Question

Does RT^2_k imply PRT^2_{k+1}?

Theorem (Flood)

PRT^2_k *does not imply* RT^3_2 *over* RCA_0 *for any* k*.*

Intuitions and formal systems

Church-Turing Thesis

For any set $A \subseteq \mathbb{N}$, the following are equivalent:

1. A is intuitively computable,
2. A is computable by a Turing machine, and
3. A is recursive.

Intuitions and formal systems

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Definition

Given $A, B \subseteq \mathbb{N}$, A is *computable from B*
if there is an Oracle Turing Machine
that computes A using B as an oracle.

Reverse Mathematics

In Reverse Mathematics,

- we formalize theorems in second order arithmetic (writing them using symbols from arithmetic, set quantifiers, and number quantifiers),
- we fix a base system of axioms called RCA_0. RCA_0 corresponds to "computable methods."

Recursive Comprehension Axiom and intuitions

Suppose that T_1 and T_2 are theorems of second order arithmetic.

T_1 implies T_2 over RCA_0
if there is a proof of T_2
using only T_1 and the axioms of RCA_0.

This corresponds to the intuition:

T_1 implies T_2 over RCA_0
if we can prove T_2
using only T_1 along with computable methods.

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PRT^2_k *does not imply* RT^3_2 *over* RCA_0 *for any* k*.*

Why is PRT^2 special?

"Direct" Proofs:

- $\mathrm{RT}^1 \implies \mathrm{PRT}^1$,
- $\mathrm{PRT}^n \implies \mathrm{RT}^n$ for any n.

"Indirect" Proofs:

- $\mathrm{RT}^n \implies \mathrm{PRT}^n$ for $n \geq 3$.

This "indirect" proof does *not* work for RT^2.

Arithmetical Comprehension Axiom

A statement is *arithmetical* if it can be written using only number quantifiers.

Example: There are infinitely many primes,
$(\forall x)(\exists y)[y > x$ and y is prime$]$.

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ACA_0 is (roughly) RCA_0 plus axioms that allow you build sets by asking *arithmetical* questions about some fixed oracle.

Example: ACA_0 implies RT^n_k for each n, k.

The indirect proof of PRT^3_k from RT^3_k

Theorem (Jockusch)

RT^n_k *implies* ACA_0 *for each* $n \geq 3$ *and* $k \geq 2$.

Theorem (Flood)

ACA_0 *implies* PRT^n_k *for each* n, k.

(I give a proof of PRT^n_k *that only asks arithmetical questions.)*

Corollary

RT^n_k *implies* PRT^n_k *for each* $n \geq 3$ *and* $k \geq 2$.

PRT^2 and RT^2

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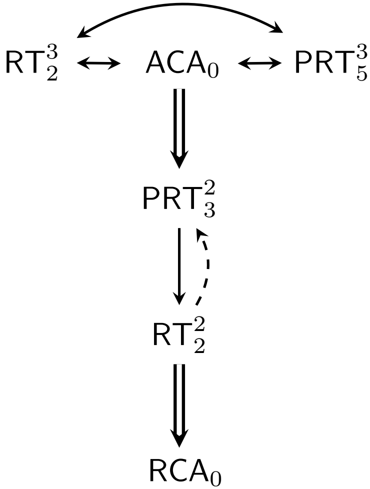
Theorem (Flood)

If f *and* ϕ *as in* PRT^2_k *are computable with* ϕ *non-decreasing, there is a* low_2 *set that is packed and* 2^{n-1}*-colored.*



Key:

$\Longrightarrow$ strict,

$\dashrightarrow$ open.

Konig's lemma for binary trees

Definition (WKL_0)

If T is an infinite binary tree,
there is an infinite path p through T.

WKL_0 and RT^2_k are both strictly in between ACA_0 and RCA_0.

Theorem (Hirst)

WKL_0 *does not imply* RT^2_k.

Corollary

WKL_0 *does not imply* PRT^2_k.

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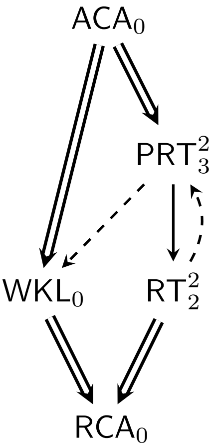
Theorem (Liu)

RT^2_k *does not imply* WKL_0.

Question

Does PRT^2 imply WKL_0?



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References

Erdős, Paul and Galvin, Fred. Some Ramsey-type theorems. *Discrete Mathematics* 87:261-269, 1991.