Kolmogorov complexity and Fourier aspects of Brownian motion

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Some History

Countable dense random SETS of reals arise naturally in

- the theory of Brownian motion –
- non-classical noises – Arveson, Tsirelson, Warren
- Fourier analysis (sets of multiplicity) – Salem, Meyer, F
- Algorithmically random Brownian motion – F
- Singular spaces in descriptive set theory – Kechris

for example.

- The study of countable dense random brings one in contact with studying random processes over spaces which are not even Polish.
- Probability theory over orbit spaces under the action of the group S_∞, which is the symmetry group of a countable set, on the space of all injections of $\mathbb{N}$ into $[0, 1]$ is required.
- These are examples of what Kechris (1999) referred to as singular spaces of Borel cardinality F_2.

- Tsirelson (2004-2006) developed a very powerful approach to random processes over these singular spaces implying that the Kechris F_2-singularity manifests in very concrete and interesting statistical properties of countable dense random sets and new aspects of Brownian motion.
- These results can be used to construct interesting sets (from the viewpoint of Fourier analysis) from Brownian motion and its effective (algorithmically random) versions (F 2012).

Random sequences

We denote by $(0,1)^\infty$ the Borel space consisting of the product of countably many copies of the unit interval and with Borel structure being given by the natural product structure which is induced by the standard Borel structure on the unit interval.
We write $(0,1)^\infty_{\neq}$ for the Borel subspace consisting of the infinite sequences in the unit interval which are pairwise distinct.

$$(0,1)^\infty_{\neq} = Inj(\mathbb{N},(0,1)) \subset (0,1)^{\mathbb{N}}.$$

Random sets from random sequences

We write S_∞ for the symmetric group of a countable set . We place on S_∞ the pointwise convergence topology thus giving S_∞ the subspace topology under its embedding into the Baire space $\mathbb{N}^{\mathbb{N}}$. The group S_∞ acts naturally (and continuously) on $(0,1)^\infty_{\neq}$:

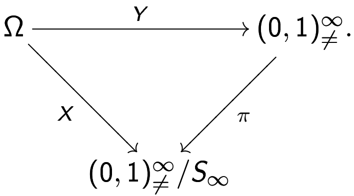
$$\sigma.(u_j : j \geq 1) := (u_{\sigma^{-1}(j)} : j \geq 1),$$

for all $(u_j) \in (0,1)^\infty_{\neq}$ and $\sigma \in S_\infty$.
The orbit space under this action is denoted by $(0,1)^\infty_{\neq}/S_\infty$.
The Borel structure on this space is given by the topology induced by the canonical mapping

$$\pi : (0,1)^\infty_{\neq} \longrightarrow (0,1)^\infty_{\neq}/S_\infty.$$

Strongly random sets

Let Ω be a standard Borel space. A *strongly* countable set in the unit interval is a measurable mapping $X : \Omega \to (0,1)^{\infty}_{\neq}/S_{\infty}$ that factors through some (traditional) random sequence Y as shown:



One can think of X as a random countable *set* induced via S_{∞}-equivalence, by a random *sequence* Y, both in the unit interval. Denote the Borel space $(0,1)^{\infty}_{\neq}/S_{\infty}$ by $CS(0,1)$.

Statistically similar random sets

For standard measure spaces (Ω_1, P_1) and (Ω_2, P_2), let there be some P_i-measurable strongly random variable $X_i : \Omega_i \to CS(0,1)$ such that the induced probability distributions on $CS(0,1)$ are the same.

We say in this case that the strongly random sets X_1 and X_2 are *statistically similar* relative to the probabilities P_1, P_2 and we write $X_1 \sim X_2$.

Generic random sets

A strongly random countable set $X : \Omega \to CS(0,1)$ is said to be *generic* relative to a probability measure P on Ω if the following is true:

> *If B is a Borel subset of the unit interval such that $\lambda(B) > 0$, then P – almost surely, $B \cap X \neq \emptyset$. On the other hand, if $\lambda(B) = 0$, then P-almost surely, $B \cap X = \emptyset$. Equivalently, if C is a Borel set such that $\lambda(C) = 1$, then, almost surely, $X \subset C$.*

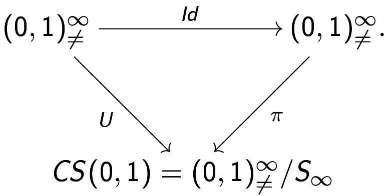
λ is the Lebesgue measure on the unit interval.

The uniform random set

Write λ^∞ for the product measure on $(0,1)^\infty$ which is the countable product of the Lebesgue measure λ on the unit interval and write Λ for the measure on $CS(0,1)$ which is the pushout of λ^∞ under π. In other words, for a Borel subset Σ of $CS(0,1)$,

$$\Lambda(\Sigma) = \lambda^\infty(\pi^{-1}\Sigma).$$

Write $U : (0,1)^\infty \to CS(0,1)$ for the strictly random set as defined by the following commutative diagram:



Then U is almost surely dense and generic relative to λ^∞. In statistics U is a model of an unordered uniform infinite sample. Moreover, it follows from the Hewitt-Savage theorem, that for every Borel subset Σ of $CS(0,1)$, it is the case that

$$\Lambda(\Sigma) \in \{0,1\}. \tag{1}$$

Note that Λ is non-atomic. Consequently, $CS(0,1)$ not a Polish space!

U is the *uniform random set*.

Local minimizers of Brownian motion

If X is a continuous function on the unit interval, then a *local minimizer* of X is a point t such that there is some closed interval $I \subset [0, 1]$ containing t such that the function X assumes a minimum value on I at the point t. We denote by $MIN(X)$ the set of local minimizers of X.

It is well-known that if X is a continuous version of Brownian motion on the unit interval, then $MIN(X)$ is almost surely a dense and countable set and that all the local minimizers of X are *strict*. This means that, for each closed subinterval I of the closed unit interval, there is a unique $\nu \in I$ where the minimum of X on I is assumed.

This has the implication that there is a subset Ω_0 of $C[0,1]$ of full Wiener measure such that one can define a measurable mapping $min : C[0,1] \supset \Omega_0 \longrightarrow (0,1)^{\infty}_{\neq}$ in such a way that the composition of min with the projection π will define a mapping $X \mapsto MIN(X)$. In the sequel this strongly random set will be denoted by MIN. To summarise, we have the following commutative diagram:

$$\begin{array}{ccc} C[0,1] \supset \Omega_0 & \xrightarrow{\text{min}} & (0,1)^{\infty}_{\neq}. \\ & \text{MIN} \searrow \quad \swarrow \pi & \\ & (0,1)^{\infty}_{\neq}/S_{\infty} & \end{array}$$

Constructing generic countable sets from Brownian motion

The next theorem says essentially that the local minimizers of a Brownian motion form a generic countable dense random set relative to the Wiener measure.

Theorem

(Tsirelson 2006.) If X is a continuous version of Brownian motion on the unit interval and B is a Borel subset of the unit interval such that $\lambda(B) > 0$, then almost surely, $B \cap MIN(X) \neq \emptyset$. On the other hand, if $\lambda(B) = 0$, then almost surely, $B \cap MIN(X) = \emptyset$. In particular, if $\lambda(C) = 1$, then, almost surely, $MIN(X) \subset C$.

Aim of project

It follows (non-trivially) that

$$MIN \sim U. \tag{2}$$

One of the aims of this research project is to explicate this statistical similarity from the viewpoint of algorithmic randomness and to bring forth distinctions between MIN and U in this manner.

Along these lines we will find explicit construction of random objects (sets of multiplicity in Fourier analysis, noises) in terms of Kolmogorov complexity.

Link with Fourier analysis

Let (m_k) be any random enumeration of the local minimizers of a continuous version of Brownian motion in the unit interval .

Let $q > 2$ and set

$$s_k = \frac{(1+m_k)}{(k!)^q}, \quad k \geq 1,$$

$$E = \{\sum_{k\geq 1} \epsilon_k s_k : \epsilon_k \in \{0,1\} \text{ all } k\}.$$

Theorem

(F 2012) The sequence (s_k) is (strongly) linearly independent over $\mathbb{Q}$ and the set E carries a non-zero Schwartz distribution whose Fourier transform vanishes at infinity.

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The second clause of the theorem can be rephrased as:

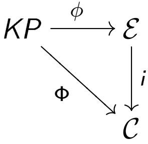
- E is a set of multiplicity

This means that there is a formal trigonometric series

$$S \sim \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k\theta}$$

with some c_k being non-zero such that the series converges to 0 everywhere outside E.

Link with Kolmogorov Complexity



(F. Adv Math 2000.)

- $KP = ML_\lambda$, λ being Lebesgue measure on $\{0,1\}^{\mathbb{N}}$
- $\mathcal{C} = ML_W$, W being Wiener measure on $C[0,1]$ (complex oscillations: Asarin Prokovskiy (1986-1988), F (2000), Kjos-Hansen Szabados (2011).)
- $\mathcal{E} \subset \{0,1\}^{\mathbb{N}^2}$ encoded versions of elements of elements of $\mathcal{C}$
- ϕ a canonical recursive isomorphism, i the natural bijection (nowhere continuous)
- $\Phi = i\phi$ measure-preserving, Borel isomorphism

Notation: $\mathcal{C}$ = complex oscillations.
Let $\mathcal{N}$ be the function that associates with every $x \in \mathcal{C}$, the set of local minimizers of x. Thus $\mathcal{N}$ is the restriction of MIN to $\mathcal{C}$. We then define the function

$$\mathcal{MIN} : KP \longrightarrow (0,1)^{\infty}_{\neq}/S_{\infty}$$

by

$$\alpha \mapsto MIN(\Phi(\alpha));$$

this means that the diagram

$$\begin{array}{ccc} KP & \xrightarrow{\mathcal{MIN}} & (0,1)^{\infty}_{\neq}/S_{\infty} \\ & \searrow^{\Phi} \quad \nearrow_{\mathcal{N}} & \\ & \mathcal{C} & \end{array}$$

commutes.

It follows from the fact that Φ is measure-preserving, that, for every Borel subset Σ of $(0,1)^{\infty}_{\neq}/S_{\infty}$:

$$\lambda(\alpha \in KP : \mathcal{MIN}(\alpha) \in \Sigma) \in \{0,1\}.$$

What is essentially at stake here is the Hewitt-Savage theorem together with the statistical similarity of three strongly random sets:

$$U \sim \mathcal{N} \sim \mathcal{MIN}.$$

Remark. It would be interesting to better understand the Borel subsets Σ of $(0,1)^{\infty}_{\neq}/S_{\infty}$ having Λ measure one such that $\mathcal{MIN}(\alpha) \in \Sigma$ for all $\alpha \in KP$.

Computing local minimizers

Theorem

(F 2012) There is a uniform algorithmic procedure that, relative to a given $\alpha \in KP$, will yield, for any closed dyadic subinterval I of the unit interval, a sequence $t_1, t_2, \ldots$ of rationals in I that converges to the (unique) local minimizer of the complex oscillation, $\Phi(\alpha)$, in I. Moreover all the local minimizers of a complex oscillation are non-computable real numbers.

Remarks and open problems

George Davie and I have recently shown that the rate of convergence of the sequence (t_k) can be effectively determined from an upper bound on the incompressibility coefficient of $\alpha \in KP$.

Whether the local minimizers can be computed from α alone is an open problem. Another open problem is whether or not the local minimizers are all in KP.

Conjecture

The local minimizers of a complex oscillation are linearly independent over the field $\mathbb{R}_r$ of recursive real numbers.

This is literally almost surely true!

Description of all local minimizers

The algorithmic version of *MIN* has a clear description:

Corollary

There is a Σ^0_4 *predicate* $C(\alpha, \nu)$ *over* $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ *such that for* $\alpha, \nu \in \{0,1\}^{\mathbb{N}}$*:*

$$C(\alpha, \nu) \iff \nu \in MIN(\Phi(\alpha)) \wedge \alpha \in KP.$$

Another link with Fourier analysis

Definition

Let α be a real such that $0 < \alpha < 1$. A sequence $\mathbf{a} = (a_k)_{k \geq 0}$ of positive real numbers is said to be α-thin if

- For each $m \geq 1$

$$\liminf_{n \to \infty} 2^{nm} a_n^{\alpha} = 0,$$

  moreover $0 < a_0 \leq \frac{1}{2}$;

- For all n

$$\Sigma_{k>n} a_k < a_n.$$

With an α-thin $\mathbf{a}$, we associate the set

$$A := A(\mathbf{a}) = \{a_0 + \Sigma_{k=1}^{\infty} \epsilon_k a_k : \epsilon_k = \pm 1, \text{ all } k\}.$$

Theorem

(F 2012) Let $\mathbf{a}$ *be an effective sequence in the unit interval which is* α*-thin for some computable* $0 < \alpha < \frac{1}{2}$*. If* x *is a complex oscillation, then the image* $x(A(\mathbf{a}))$ *of the set* $A(\mathbf{a})$ *under* x *will be linearly independent over the field* $\mathbb{R}_r$ *of computable real numbers. Moreover, the image* $x(A(\mathbf{a}))$ *will contain no computable real numbers.*

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