Ergodic theory and strong randomness notions

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Central question

- Random points are regular with respect to measure.
- Ergodic theorems say that the orbits of certain transformations have regularity properties for almost every point.

How are these two kinds of regularity related?

Basic environment

- We work in the Cantor space: $\{0,1\}^\omega$.
- The basic open sets take the form $[\sigma]$: the set of all infinite binary sequences extending the finite string σ.
- The measure of a basic open set $[\sigma]$ is $2^{-|\sigma|}$.

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Imagine a neon atom at a point x in a sealed box with a force acting on it every second that moves it to $T(x)$, so $T^n(x)$ gives the position of the atom after n seconds.

Ergodic theorems lets us describe the movement of the atom over time via properties of the set

$$\{x, T(x), T^2(x), T^3(x), \ldots\}.$$

More formally...

We work in a probability space (X, μ) and consider a function $T : X \to X$ and a measurable $A \subseteq X$.

- T is *measure preserving* if $\mu(T^{-1}(A)) = \mu(A)$.
- A is T-*invariant* if $T^{-1}(A) = A$ up to a null set.
- T is *ergodic* if it is measure preserving and every T-invariant measurable subset has either measure 0 or measure 1.

Birkhoff's ergodic theorem

Theorem

Let (X, μ) be a probability space, let $T : X \to X$ be ergodic, and let $f \in L^1(X)$. Then for almost all $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} f(T^i(x)) = \int f \, d\mu.$$

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If f is the characteristic function of a measurable set A, this becomes

$$\lim_{n\to\infty} \frac{|\{i \mid i < n \text{ and } T^i(x) \in A\}|}{n} = \mu(A).$$

Birkhoff points

Let (X, μ) be a probability space, let $T : X \to X$ be a measure-preserving function, and let $\mathcal{F}$ be a collection of functions in $L_1(X)$.

Definition

- A point $x \in X$ is a *weak Birkhoff point* for T with respect to $\mathcal{F}$ if for every $f \in \mathcal{F}$,

$$\lim_n \frac{1}{n} \sum_{i<n} f(T^i(x))$$

  converges.
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Summary of the ergodic case

- A point is Martin-Löf random if and only if it is a Birkhoff point for ergodic maps with respect to effectively closed sets.
- A point is Schnorr random if and only if it is a Birkhoff point for ergodic maps with respect to effectively closed sets with computable measure.

The nonergodic case

Theorem (V'yugin, 1997)

If $x \in \{0,1\}^\omega$ *is Martin-Löf random, then for any computable function* f *and any computable measure-preserving transformation* T*, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} f(T^i(x))$$

exists.

The other direction

Theorem (F. & Towsner)

If $x \in \{0,1\}^\omega$ is not Martin-Löf random, then there is a computable function f and a computable measure-preserving transformation T such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} f(T^i(x))$$

does not exist.

Key tool: upcrossings

Suppose we have

- a measure-preserving, invertible $T : \{0,1\}^\omega \to \{0,1\}^\omega$,
- a point $x \in \{0,1\}^\omega$, and
- a measurable f.

Definition

Given two rationals $\alpha < \beta$, an *upcrossing sequence* for α, β is a sequence

$$0 \leq u_1 < v_1 < u_2 < \ldots < u_N < v_N$$

such that for all $i \leq N$,

$$\frac{1}{u_i + 1} \sum_{j=0}^{u_i} f(T^j(x)) < \alpha \quad \text{and} \quad \frac{1}{v_i + 1} \sum_{j=0}^{v_i} f(T^j(x)) > \beta.$$

Define $\tau(x, f, \alpha, \beta)$ as the supremum of the lengths of upcrossing sequences for α, β.

Construction

- Take a Martin-Löf test $\langle V_n \rangle$ witnessing that x isn't Martin-Löf random. Build a set A such that every element of $\cap V_n$ maps through A half the time and then maps through another set B to bring its average time in A down to a third infinitely often.

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- More formally: When a string σ enters some V_j, we make sure that the lower bound on $\tau(x, \chi_A, \frac{1}{3}, \frac{1}{2})$ increases for each $x \in [\sigma]$.

The lower semicomputable case

Suppose we have a sequence of uniformly computable increasing functions $\langle f_i \rangle$ converging to f.

Goal

We want to bound the number of upcrossings in f.

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Problem

$\tau(x, f_i, \alpha, \beta)$ *is not necessarily monotonic in i: an upcrossing sequence for f_i may not be an upcrossing sequence for f_{i+1}.*

A generalization of upcrossings

Suppose we have

- a measure-preserving, invertible $T : \{0,1\}^\omega \to \{0,1\}^\omega$,
- a point $x \in \{0,1\}^\omega$, and
- measurable f and h.

Definition

Given two rationals $\alpha < \beta$, a *loose upcrossing sequence* for α, β is a sequence

$$0 \leq u_1 < v_1 < u_2 < \ldots < u_N < v_N$$

such that for all $i \leq N$,

$$\frac{1}{u_i + 1} \sum_{j=0}^{u_i} f(T^j(x)) < \alpha \quad \text{and} \quad \frac{1}{v_i + 1} \sum_{j=0}^{v_i} (f + h)(T^j(x)) > \beta.$$

Define $v(x, f, h, \alpha, \beta)$ as the supremum of the lengths of loose upcrossing sequences for α, β.

A partial result

Theorem (F. and Towsner)

If x is weakly 2-random, f is lower semicomputable, and T is a computable measure-preserving transformation, then

$$\lim_n \frac{1}{n+1} \sum_{j=0}^{n} f(T^j(x))$$

converges.

Main ideas

Proof by contrapositive.

- If this limit doesn't converge, then $\tau(x, f, \alpha, \beta)$ is infinite for some $\alpha < \beta$.
- Define $V_n = \{x \mid (\exists m \geq n) v(x, f_n, f_m - f_n, \alpha, \beta) \geq n\}$.
- The V_ns are nested: if

  $$v(x, f_{n+1}, f_m - f_{n+1}, \alpha, \beta) \geq n + 1$$

  we can find a loose upcrossing sequence witnessing this, which also witnesses

  $$v(x, f_n, f_m - f_n, \alpha, \beta) \geq n + 1 > n.$$

- $\lim_n \mu([V_n]) = 0$: technical lemma.

Improvements?

Theorem (F. and Towsner)

Suppose the following is true:

> *Let f and $\epsilon > 0$ be given and let $0 \leq h_0 \leq h_1 \leq \ldots \leq h_n$ be given with $||h_n||_{L^\infty} < \epsilon$. Then*
>
> $$\int_X \sup_n \tau(x, f + h_n, \alpha, \beta)\, dx < c$$
>
> *where c is a computable bound depending on $||f||_{L^\infty}$ and ϵ.*

Then if x is balanced random and f is lower semicomputable, then $\lim_n \frac{1}{n+1} \sum_{j=0}^{n} f(T^j(x))$ converges.

Recall that a balanced test is a sequence of r.e. sets $\langle W_{f(i)} \rangle$ where f is a 2^n-r.e. function and $\mu([W_{f(i)}]) \leq 2^{-i}$ for every i.

Thank you!