(Almost) Lowness for K and Finite Self-Information

Ian Herbert

University of California, Berkeley

July 3, 2012

Lowness for K

Definition (Muchnik)

A real $A \in 2^\omega$ is called *low for K* if there is a constant c such that for all $\sigma \in 2^{<\omega}$ $K(\sigma) \leq K^A(\sigma) + c$

These reals have been well studied, and the collection of lows for K, $\mathcal{LK}$, has many nice properties. For instance:

- $\mathcal{LK}$ is closed downwards under $\leq_T$
- $\mathcal{LK}$ is closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003)
- $\mathcal{LK}$ has only countably many elements, and they are all Δ^0_2(Chaitin, 1976)
- $A \in \mathcal{LK}$ iff A is *K-trivial*: $\exists d \forall n K(A \restriction n) \leq K(n) + d$ (Nies, 2005)

Lowness for K

Definition (Muchnik)

A real $A \in 2^\omega$ is called *low for K* if there is a constant c such that for all $\sigma \in 2^{<\omega}$ $K(\sigma) \leq K^A(\sigma) + c$

These reals have been well studied, and the collection of lows for K, $\mathcal{LK}$, has many nice properties. For instance:

- $\mathcal{LK}$ is closed downwards under $\leq_T$
- $\mathcal{LK}$ is closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003)
- $\mathcal{LK}$ has only countably many elements, and they are all Δ^0_2(Chaitin, 1976)
- $A \in \mathcal{LK}$ iff A is *K-trivial*: $\exists d \forall n K(A \upharpoonright n) \leq K(n) + d$ (Nies, 2005)

Definition (Muchnik)

A real $A \in 2^\omega$ is called *low for K* if there is a constant c such that for all $\sigma \in 2^{<\omega}$ $K(\sigma) \leq K^A(\sigma) + c$

These reals have been well studied, and the collection of lows for K, $\mathcal{LK}$, has many nice properties. For instance:

- $\mathcal{LK}$ is closed downwards under $\leq_T$
- $\mathcal{LK}$ is closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003)
- $\mathcal{LK}$ has only countably many elements, and they are all Δ^0_2(Chaitin, 1976)
- $A \in \mathcal{LK}$ iff A is *K-trivial*: $\exists d \forall n K(A \upharpoonright n) \leq K(n) + d$ (Nies, 2005)

Lowness for K

Definition (Muchnik)

A real $A \in 2^\omega$ is called *low for K* if there is a constant c such that for all $\sigma \in 2^{<\omega}$ $K(\sigma) \leq K^A(\sigma) + c$

These reals have been well studied, and the collection of lows for K, $\mathcal{LK}$, has many nice properties. For instance:

- $\mathcal{LK}$ is closed downwards under $\leq_T$
- $\mathcal{LK}$ is closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003)
- $\mathcal{LK}$ has only countably many elements, and they are all Δ^0_2(Chaitin, 1976)
- $A \in \mathcal{LK}$ iff A is *K-trivial*: $\exists d \forall n K(A \upharpoonright n) \leq K(n) + d$ (Nies, 2005)

Lowness for K

Definition (Muchnik)

A real $A \in 2^\omega$ is called *low for K* if there is a constant c such that for all $\sigma \in 2^{<\omega}$ $K(\sigma) \leq K^A(\sigma) + c$

These reals have been well studied, and the collection of lows for K, $\mathcal{LK}$, has many nice properties. For instance:

- $\mathcal{LK}$ is closed downwards under $\leq_T$
- $\mathcal{LK}$ is closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003)
- $\mathcal{LK}$ has only countably many elements, and they are all Δ^0_2(Chaitin, 1976)
- $A \in \mathcal{LK}$ iff A is *K-trivial*: $\exists d \forall n K(A \restriction n) \leq K(n) + d$ (Nies, 2005)

Lowness for K

Definition (Muchnik)

A real $A \in 2^\omega$ is called *low for K* if there is a constant c such that for all $\sigma \in 2^{<\omega}$ $K(\sigma) \leq K^A(\sigma) + c$

These reals have been well studied, and the collection of lows for K, $\mathcal{LK}$, has many nice properties. For instance:

- $\mathcal{LK}$ is closed downwards under $\leq_T$
- $\mathcal{LK}$ is closed under effective join (Downey, Hirschfeldt, Nies, Stephan, 2003)
- $\mathcal{LK}$ has only countably many elements, and they are all Δ^0_2(Chaitin, 1976)
- $A \in \mathcal{LK}$ iff A is *K-trivial*: $\exists d \forall n K(A \upharpoonright n) \leq K(n) + d$ (Nies, 2005)

Almost Lowness for K

We can relax these definitions slightly:

Definition

For a function $f : 2^{<\omega} \to \mathbb{N}$ a real $A \in 2^\omega$ is called *low for K up to f* if there is a constant c such that for all $\sigma \in 2^{<\omega}$

$$K(\sigma) \leq K^A(\sigma) + f(\sigma) + c$$

For a function $g : \mathbb{N} \to \mathbb{N}$, A is called *K-trivial up to g* if there is a constant d such that for all $n \in \mathbb{N}$

$$K(A \upharpoonright n) \leq K(n) + g(n) + d$$

We let $\mathcal{LK}_f$ denote the class of reals low for K up to f and $\mathcal{KT}_g$ denote the class of reals that are K-trivial up to g.

Obviously, the choice of f will affect the properties of $\mathcal{LK}_f$. For this talk, we will restrict ourselves to the case where f is an *order*, i.e. nondecreasing (for the partial ordering $\sigma <_l \tau \Leftrightarrow |\sigma| < |\tau|$) and unbounded, and most of the results mentioned will be for Δ^0_2 orders.

It follows from a result of Baartse and Barmpalias that there is a Δ^0_3 order h such that $\mathcal{LK}_h = \mathcal{LK}$.

Obviously, the choice of f will affect the properties of $\mathcal{LK}_f$. For this talk, we will restrict ourselves to the case where f is an *order*, i.e. nondecreasing (for the partial ordering $\sigma <_l \tau \Leftrightarrow |\sigma| < |\tau|$) and unbounded, and most of the results mentioned will be for Δ^0_2 orders.

It follows from a result of Baartse and Barmpalias that there is a Δ^0_3 order h such that $\mathcal{LK}_h = \mathcal{LK}$.

Almost Lowness for K is not like Lowness for K

In general, $\mathcal{LK} \neq \mathcal{LK}_f$, even for slowly growing f. In fact, for any Δ^0_2 order f:

- $\mathcal{LK}_f$ is still closed downwards under $\leq_T$
- $\mathcal{LK}_f$ is not closed under effective join; Any real is below the join of two elements of $\mathcal{LK}_f$
- $\mathcal{LK}_f$ is uncountable, so in particular there are elements that are not Δ^0_2

Also, the notions of K-triviality and lowness for K come apart in this setting.

- There is a computable order $f : 2^{<\omega} \to \mathbb{N}$ and a real A such that for all Δ^0_2 orders $g : \mathbb{N} \to \mathbb{N}$, $A \in \mathcal{KT}_g$, but $A \notin \mathcal{LK}_f$

Almost Lowness for K is not like Lowness for K

In general, $\mathcal{LK} \neq \mathcal{LK}_f$, even for slowly growing f. In fact, for any Δ^0_2 order f:

- $\mathcal{LK}_f$ is still closed downwards under $\leq_T$
- $\mathcal{LK}_f$ is not closed under effective join; Any real is below the join of two elements of $\mathcal{LK}_f$
- $\mathcal{LK}_f$ is uncountable, so in particular there are elements that are not Δ^0_2

Also, the notions of K-triviality and lowness for K come apart in this setting.

- There is a computable order $f : 2^{<\omega} \to \mathbb{N}$ and a real A such that for all Δ^0_2 orders $g : \mathbb{N} \to \mathbb{N}$, $A \in \mathcal{KT}_g$, but $A \notin \mathcal{LK}_f$

Almost Lowness for K is not like Lowness for K

In general, $\mathcal{LK} \neq \mathcal{LK}_f$, even for slowly growing f. In fact, for any Δ^0_2 order f:

- $\mathcal{LK}_f$ is still closed downwards under $\leq_T$
- $\mathcal{LK}_f$ is not closed under effective join; Any real is below the join of two elements of $\mathcal{LK}_f$
- $\mathcal{LK}_f$ is uncountable, so in particular there are elements that are not Δ^0_2

Also, the notions of K-triviality and lowness for K come apart in this setting.

- There is a computable order $f : 2^{<\omega} \to \mathbb{N}$ and a real A such that for all Δ^0_2 orders $g : \mathbb{N} \to \mathbb{N}$, $A \in \mathcal{KT}_g$, but $A \notin \mathcal{LK}_f$

Almost Lowness for K is not like Lowness for K

In general, $\mathcal{LK} \neq \mathcal{LK}_f$, even for slowly growing f. In fact, for any Δ^0_2 order f:

- $\mathcal{LK}_f$ is still closed downwards under $\leq_T$
- $\mathcal{LK}_f$ is not closed under effective join; Any real is below the join of two elements of $\mathcal{LK}_f$
- $\mathcal{LK}_f$ is uncountable, so in particular there are elements that are not Δ^0_2

Also, the notions of K-triviality and lowness for K come apart in this setting.

- There is a computable order $f : 2^{<\omega} \to \mathbb{N}$ and a real A such that for all Δ^0_2 orders $g : \mathbb{N} \to \mathbb{N}$, $A \in \mathcal{KT}_g$, but $A \notin \mathcal{LK}_f$

Almost Lowness for K is not like Lowness for K

In general, $\mathcal{LK} \neq \mathcal{LK}_f$, even for slowly growing f. In fact, for any Δ^0_2 order f:

- $\mathcal{LK}_f$ is still closed downwards under $\leq_T$
- $\mathcal{LK}_f$ is not closed under effective join; Any real is below the join of two elements of $\mathcal{LK}_f$
- $\mathcal{LK}_f$ is uncountable, so in particular there are elements that are not Δ^0_2

Also, the notions of K-triviality and lowness for K come apart in this setting.

- There is a computable order $f : 2^{<\omega} \to \mathbb{N}$ and a real A such that for all Δ^0_2 orders $g : \mathbb{N} \to \mathbb{N}$, $A \in \mathcal{KT}_g$, but $A \notin \mathcal{LK}_f$

Almost Lowness for K is not like Lowness for K

In general, $\mathcal{LK} \neq \mathcal{LK}_f$, even for slowly growing f. In fact, for any Δ^0_2 order f:

- $\mathcal{LK}_f$ is still closed downwards under $\leq_T$
- $\mathcal{LK}_f$ is not closed under effective join; Any real is below the join of two elements of $\mathcal{LK}_f$
- $\mathcal{LK}_f$ is uncountable, so in particular there are elements that are not Δ^0_2

Also, the notions of K-triviality and lowness for K come apart in this setting.

- There is a computable order $f : 2^{<\omega} \to \mathbb{N}$ and a real A such that for all Δ^0_2 orders $g : \mathbb{N} \to \mathbb{N}$, $A \in \mathcal{KT}_g$, but $A \notin \mathcal{LK}_f$

Main Theorems

Theorem (H.)

For any Δ^0_2 order f there is a perfect Π^0_1 class $\mathcal{P}$ such that $\mathcal{P} \subseteq \mathcal{LK}_f$ and for any $C \in 2^\omega$ there are $A, B \in \mathcal{P}$ such that $C \leq_T A \oplus B$.

Theorem (H.)

There is a perfect class $\mathcal{Q}$ (not Π^0_1) such that for every Δ^0_2 order f, $\mathcal{Q} \subseteq \mathcal{LK}_f$, and for any $C \in 2^\omega$ there are $A, B \in \mathcal{Q}$ such that $C \leq_T A \oplus B$.

Main Theorems

Theorem (H.)

For any Δ^0_2 order f there is a perfect Π^0_1 class $\mathcal{P}$ such that $\mathcal{P} \subseteq \mathcal{LK}_f$ and for any $C \in 2^\omega$ there are $A, B \in \mathcal{P}$ such that $C \leq_T A \oplus B$.

Theorem (H.)

There is a perfect class $\mathcal{Q}$ (not Π^0_1) such that for every Δ^0_2 order f, $\mathcal{Q} \subseteq \mathcal{LK}_f$, and for any $C \in 2^\omega$ there are $A, B \in \mathcal{Q}$ such that $C \leq_T A \oplus B$.

Proof Sketch

We want to build a computable tree T, such that the paths through T are all in $\mathcal{LK}_f$. To ensure $K(\sigma) \leq^+ K^A(\sigma) + f(\sigma)$ we build a Kraft-Chaitin set L, and enumerate requests for descriptions of σ when we see new descriptions converge on paths through T.

The problem is that as the number of branches increases, the same mass can be used to give descriptions of different σ on many branches, and we might be forced to add too much mass to L.

Proof Sketch

We want to build a computable tree T, such that the paths through T are all in $\mathcal{LK}_f$. To ensure $K(\sigma) \leq^+ K^A(\sigma) + f(\sigma)$ we build a Kraft-Chaitin set L, and enumerate requests for descriptions of σ when we see new descriptions converge on paths through T.

The problem is that as the number of branches increases, the same mass can be used to give descriptions of different σ on many branches, and we might be forced to add too much mass to L.

Proof Sketch

So, we use the fact that we are only paying up to a factor of f. We know f is an order, so for any i, only finitely many σ have $f(\sigma) = i$. We contrive to make sure that all descriptions of σ with $f(\sigma) = i$ converge on T below the level where T branches for the $i+1$st time.

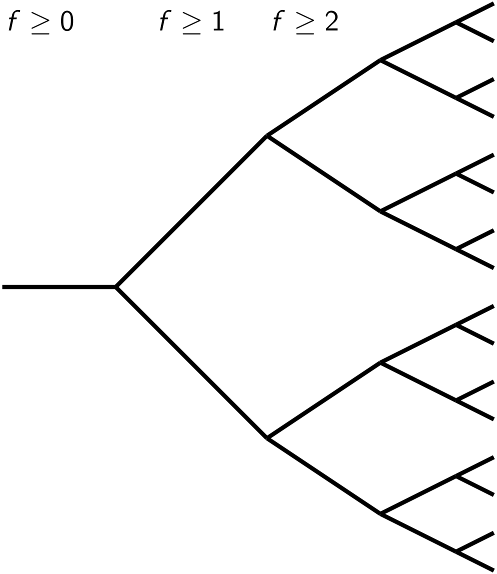
If it looks like this is failing for a given i, keep the path above the $i+1$st branching level with the most mass (and all identical extensions from nodes at that level), kill all other paths, and move the branching level up (picture to come).

Proof Sketch

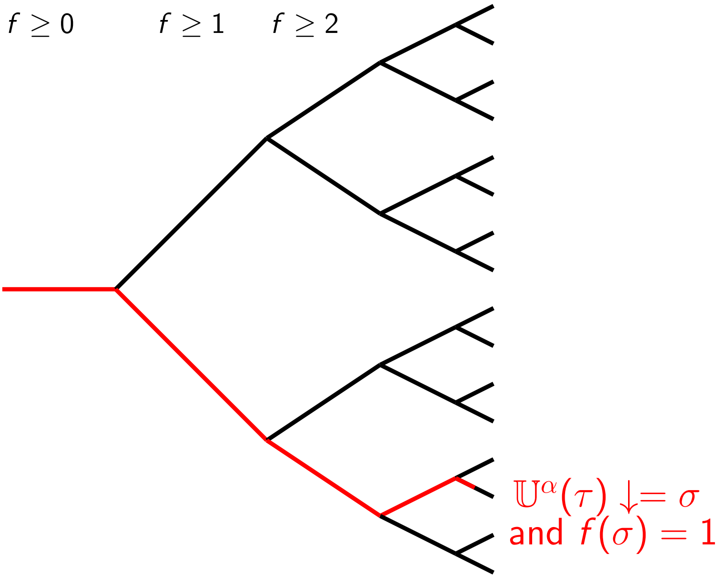
So, we use the fact that we are only paying up to a factor of f. We know f is an order, so for any i, only finitely many σ have $f(\sigma) = i$. We contrive to make sure that all descriptions of σ with $f(\sigma) = i$ converge on T below the level where T branches for the $i + 1$st time.

If it looks like this is failing for a given i, keep the path above the $i + 1$st branching level with the most mass (and all identical extensions from nodes at that level), kill all other paths, and move the branching level up (picture to come).

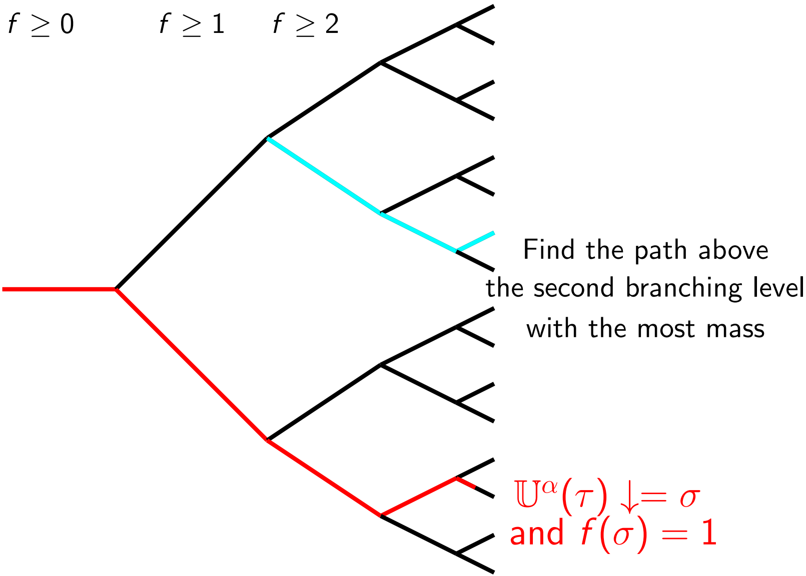
Picture



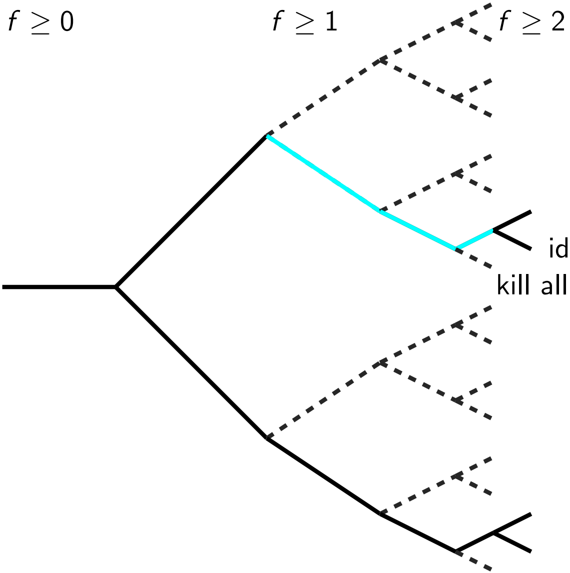
Picture


$f \geq 0$
$f \geq 1$
$f \geq 2$
$\mathbb{U}^{\alpha}(\tau)\downarrow = \sigma$
and $f(\sigma) = 1$

Picture

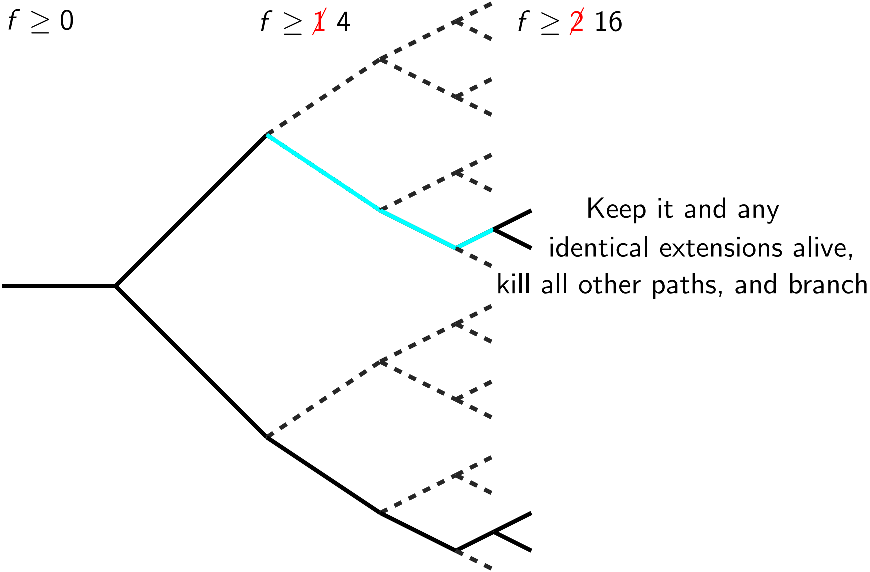

$f \geq 0$
$f \geq 1$
$f \geq 2$
Find the path above
the second branching level
with the most mass
$\mathbb{U}^{\alpha}(\tau)\downarrow = \sigma$
and $f(\sigma) = 1$

Picture



Keep it and any identical extensions alive, kill all other paths, and branch

Picture


$f \geq 0$
$f \geq \not{1}\ 4$
$f \geq \not{2}\ 16$
Keep it and any
identical extensions alive,
kill all other paths, and branch

The harder case

To build a tree that works for all f is more complicated. We don't know beforehand which approximations $\phi_{e,s}$ actually correspond to orders. We have to make one tree and use the $(2e)$th branchings to guess whether or not $\phi_{e,s}$ gives an order.

The tree we want will then be the subtree that gets all these guesses right, and, by virtue of being right, it will only be injured finitely often.

Changing Gears - Mutual Information

Definition (Levin)

The *mutual information* of reals A and B is

$$I(A:B) = \log \sum_{\sigma,\tau \in 2^{<\omega}} 2^{K(\sigma)-K^A(\sigma)+K(\tau)-K^B(\tau)-K(\sigma,\tau)}.$$

One way to think of this is to view $K(\sigma) - K^A(\sigma)$ as a measure of how much A 'knows' about σ.

Changing Gears - Mutual Information

Definition (Levin)

The *mutual information* of reals A and B is

$$I(A:B) = \log \sum_{\sigma,\tau \in 2^{<\omega}} 2^{K(\sigma)-K^A(\sigma)+K(\tau)-K^B(\tau)-K(\sigma,\tau)}.$$

One way to think of this is to view $K(\sigma) - K^A(\sigma)$ as a measure of how much A 'knows' about σ.

Definition (Levin)

The *mutual information* of reals A and B is

$$I(A:B) = \log \sum_{\sigma,\tau \in 2^{<\omega}} 2^{K(\sigma)-K^A(\sigma)+K(\tau)-K^B(\tau)-K(\sigma,\tau)}.$$

One way to think of this is to view $K(\sigma) - K^A(\sigma)$ as a measure of how much A 'knows' about σ.

Self-Information

Definition

We say A has *finite self-information* if $I(A : A) < \infty$.

Observation

If A is low for K, then A has finite self-information.

If A is MLR, then A has infinite self-information.[Levin]

Self-Information

Definition

We say A has *finite self-information* if $I(A : A) < \infty$.

Observation

If A is low for K, then A has finite self-information.

If A is MLR, then A has infinite self-information.[Levin]

Self-Information

Definition

We say A has *finite self-information* if $I(A : A) < \infty$.

Observation

If A is low for K, then A has finite self-information.

If A is MLR, then A has infinite self-information.[Levin]

Self-Information

Proposition (Hirschfeldt, Reimann, Weber)

In fact, A is low for K iff $I(A : B) < \infty$ for all B.

Proposition (Hirschfeldt, Reimann, Weber)

In fact, A is low for K iff $I(A : B) < \infty$ for all B.

In the forward direction the proof uses symmetry of information for finite strings [Levin, Gács].

Proposition (Hirschfeldt, Reimann, Weber)

In fact, A is low for K iff $I(A : B) < \infty$ for all B.

In the forward direction the proof uses symmetry of information for finite strings [Levin, Gács].In the reverse direction, use Low for K $\Leftrightarrow$ Low for MLR and let $B \in \mathrm{MLR}/\mathrm{MLR}^A$. By the Ample Excess Lemma of Miller and Yu, the mutual information of A and B about the initial segments of B will be infinite (this proof is due to Day).

Finite Self-Information

There was the question of whether the reals that are low for K were exactly the reals with finite self-information. Hirschfeldt and Weber answered in the negative.

Theorem (Hirschfeldt, Weber)

There is a c.e. A with finite self-information that is not low for K

They left open the question as to whether the reals with finite self-information were all Δ_2^0.

Finite Self-Information

There was the question of whether the reals that are low for K were exactly the reals with finite self-information. Hirschfeldt and Weber answered in the negative.

Theorem (Hirschfeldt, Weber)

There is a c.e. A with finite self-information that is not low for K

They left open the question as to whether the reals with finite self-information were all Δ^0_2.

Finite Self-Information

There was the question of whether the reals that are low for K were exactly the reals with finite self-information. Hirschfeldt and Weber answered in the negative.

Theorem (Hirschfeldt, Weber)

There is a c.e. A with finite self-information that is not low for K

They left open the question as to whether the reals with finite self-information were all Δ^0_2.

Self-Information

In the proof they construct a Δ^0_2 function f_{HW} such that

$$\sum_{\sigma,\tau\in 2^{<\omega}} 2^{f_{HW}(\sigma)+f_{HW}(\tau)-K(\sigma,\tau)} < \infty,$$

and then build a non-low for K real that is low for K up to f_{HW}.

It turns out f_{HW} is close enough to being an order that our theorem applies, so we can get:

Corollary

There is a perfect Π^0_1 class of reals that have finite self-information, and any real is below the join of two reals with finite self-information.

Self-Information

In the proof they construct a Δ^0_2 function f_{HW} such that

$$\sum_{\sigma,\tau \in 2^{<\omega}} 2^{f_{HW}(\sigma)+f_{HW}(\tau)-K(\sigma,\tau)} < \infty,$$

and then build a non-low for K real that is low for K up to f_{HW}.

It turns out f_{HW} is close enough to being an order that our theorem applies, so we can get:

Corollary

There is a perfect Π^0_1 class of reals that have finite self-information, and any real is below the join of two reals with finite self-information.

Self-Information

In the proof they construct a Δ^0_2 function f_{HW} such that

$$\sum_{\sigma,\tau \in 2^{<\omega}} 2^{f_{HW}(\sigma)+f_{HW}(\tau)-K(\sigma,\tau)} < \infty,$$

and then build a non-low for K real that is low for K up to f_{HW}.

It turns out f_{HW} is close enough to being an order that our theorem applies, so we can get:

Corollary

There is a perfect Π^0_1 class of reals that have finite self-information, and any real is below the join of two reals with finite self-information.

What this gives us

First, it tells us about finite self-information.

- We may want to use a different definition of mutual information, if so many reals have finite self-information under this one.
- Or, we may not. Maybe we can accept that reals might have information (in that they are not low for K), but might not know what information they have. In general, it is Δ^0_2 in A to tell for which σ $K(\sigma) - K^A(\sigma)$ is large.

Second, it tells us about $\mathcal{LK}_f$. Although $\mathcal{LK}_f$ doesn't behave as nicely as $\mathcal{LK}$, it is possible for certain f's to get all the elements of $\mathcal{LK}_f$ to be nice. All the elements of $\mathcal{LK}_{f_{HW}}$ are incomplete (none can compute an MLR) and in fact they are all GL_1 [Hirschfeldt, Weber].

What this gives us

First, it tells us about finite self-information.

- We may want to use a different definition of mutual information, if so many reals have finite self-information under this one.
- Or, we may not. Maybe we can accept that reals might have information (in that they are not low for K), but might not know what information they have. In general, it is Δ^0_2 in A to tell for which σ $K(\sigma) - K^A(\sigma)$ is large.

Second, it tells us about $\mathcal{LK}_f$. Although $\mathcal{LK}_f$ doesn't behave as nicely as $\mathcal{LK}$, it is possible for certain f's to get all the elements of $\mathcal{LK}_f$ to be nice. All the elements of $\mathcal{LK}_{f_{HW}}$ are incomplete (none can compute an MLR) and in fact they are all GL_1 [Hirschfeldt, Weber].

What this gives us

First, it tells us about finite self-information.

- We may want to use a different definition of mutual information, if so many reals have finite self-information under this one.
- Or, we may not. Maybe we can accept that reals might have information (in that they are not low for K), but might not know what information they have. In general, it is Δ^0_2 in A to tell for which σ $K(\sigma) - K^A(\sigma)$ is large.

Second, it tells us about $\mathcal{LK}_f$. Although $\mathcal{LK}_f$ doesn't behave as nicely as $\mathcal{LK}$, it is possible for certain f's to get all the elements of $\mathcal{LK}_f$ to be nice. All the elements of $\mathcal{LK}_{f_{HW}}$ are incomplete (none can compute an MLR) and in fact they are all GL_1 [Hirschfeldt, Weber].

What this gives us

First, it tells us about finite self-information.

- We may want to use a different definition of mutual information, if so many reals have finite self-information under this one.
- Or, we may not. Maybe we can accept that reals might have information (in that they are not low for K), but might not know what information they have. In general, it is Δ^0_2 in A to tell for which σ $K(\sigma) - K^A(\sigma)$ is large.

Second, it tells us about $\mathcal{LK}_f$. Although $\mathcal{LK}_f$ doesn't behave as nicely as $\mathcal{LK}$, it is possible for certain f's to get all the elements of $\mathcal{LK}_f$ to be nice. All the elements of $\mathcal{LK}_{f_{HW}}$ are incomplete (none can compute an MLR) and in fact they are all GL_1 [Hirschfeldt, Weber].

What this gives us

First, it tells us about finite self-information.

- We may want to use a different definition of mutual information, if so many reals have finite self-information under this one.
- Or, we may not. Maybe we can accept that reals might have information (in that they are not low for K), but might not know what information they have. In general, it is Δ^0_2 in A to tell for which σ $K(\sigma) - K^A(\sigma)$ is large.

Second, it tells us about $\mathcal{LK}_f$. Although $\mathcal{LK}_f$ doesn't behave as nicely as $\mathcal{LK}$, it is possible for certain f's to get all the elements of $\mathcal{LK}_f$ to be nice. All the elements of $\mathcal{LK}_{f_{HW}}$ are incomplete (none can compute an MLR) and in fact they are all GL_1 [Hirschfeldt, Weber].

What this gives us

First, it tells us about finite self-information.

- We may want to use a different definition of mutual information, if so many reals have finite self-information under this one.
- Or, we may not. Maybe we can accept that reals might have information (in that they are not low for K), but might not know what information they have. In general, it is Δ^0_2 in A to tell for which σ $K(\sigma) - K^A(\sigma)$ is large.

Second, it tells us about $\mathcal{LK}_f$. Although $\mathcal{LK}_f$ doesn't behave as nicely as $\mathcal{LK}$, it is possible for certain f's to get all the elements of $\mathcal{LK}_f$ to be nice. All the elements of $\mathcal{LK}_{f_{HW}}$ are incomplete (none can compute an MLR) and in fact they are all GL_1 [Hirschfeldt, Weber].

What this gives us

First, it tells us about finite self-information.

- We may want to use a different definition of mutual information, if so many reals have finite self-information under this one.
- Or, we may not. Maybe we can accept that reals might have information (in that they are not low for K), but might not know what information they have. In general, it is Δ^0_2 in A to tell for which σ $K(\sigma) - K^A(\sigma)$ is large.

Second, it tells us about $\mathcal{LK}_f$. Although $\mathcal{LK}_f$ doesn't behave as nicely as $\mathcal{LK}$, it is possible for certain f's to get all the elements of $\mathcal{LK}_f$ to be nice. All the elements of $\mathcal{LK}_{f_{HW}}$ are incomplete (none can compute an MLR) and in fact they are all GL_1 [Hirschfeldt, Weber].

Other applications

Definition

(Mayordomo) The *effective Hausdorff dimension* of a real S is

$$\dim(S) = \liminf_{n\to\infty} \frac{K(S \upharpoonright n)}{n}$$

(Athreya, Hitchcock, Lutz, and Mayordomo) The *effective packing dimension* of a real S is

$$\mathrm{Dim}(S) = \limsup_{n\to\infty} \frac{K(S \upharpoonright n)}{n}$$

Definition

A real A is *low for effective Hausdorff dimension* if for every real S $\dim(S) = \dim^A(S)$.
A real A is *low for effective packing dimension* if for every real S $\mathrm{Dim}(S) = \mathrm{Dim}^A(S)$.

Other applications

Definition

(Mayordomo) The *effective Hausdorff dimension* of a real S is

$$\dim(S) = \liminf_{n\to\infty} \frac{K(S \upharpoonright n)}{n}$$

(Athreya, Hitchcock, Lutz, and Mayordomo) The *effective packing dimension* of a real S is

$$\mathrm{Dim}(S) = \limsup_{n\to\infty} \frac{K(S \upharpoonright n)}{n}$$

Definition

A real A is *low for effective Hausdorff dimension* if for every real S $\dim(S) = \dim^A(S)$.
A real A is *low for effective packing dimension* if for every real S $\mathrm{Dim}(S) = \mathrm{Dim}^A(S)$.

Other applications

Observation (Hirschfeldt, Weber)

If A is low for K up to $\log|\sigma|$, then A is low for effective Hausdorff dimension and effective packing dimension.

$\log|\sigma|$ is a Δ^0_2 order, so

Corollary (independently by Lempp, Miller, Ng, Turetsky, and Weber)

There is a perfect Π^0_1 class of reals that are low for both notions of effective dimension, and any real is below the join of two such reals.

Other applications

Observation (Hirschfeldt, Weber)

If A is low for K up to $\log|\sigma|$, then A is low for effective Hausdorff dimension and effective packing dimension.

$\log|\sigma|$ is a Δ^0_2 order, so

Corollary (independently by Lempp, Miller, Ng, Turetsky, and Weber)

There is a perfect Π^0_1 class of reals that are low for both notions of effective dimension, and any real is below the join of two such reals.

Other applications

Observation (Hirschfeldt, Weber)

If A is low for K up to $\log|\sigma|$, then A is low for effective Hausdorff dimension and effective packing dimension.

$\log|\sigma|$ is a Δ^0_2 order, so

Corollary (independently by Lempp, Miller, Ng, Turetsky, and Weber)

There is a perfect Π^0_1 class of reals that are low for both notions of effective dimension, and any real is below the join of two such reals.

The end

Thanks!