On the inversion of computable functions

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Shift-invariant measures

- A probability measure P on $2^{\mathbb{N}}$ is ***shift-invariant*** if

$$P[w] = P[0w] + P[1w],$$

- if P is shift-invariant then P-almost surely, the frequency of occurrences of every $w \in 2^{<\mathbb{N}}$ converge [Birkhoff, 1931],
- P is ***ergodic*** if the limit frequencies do not depend on x,
- P is ergodic iff it cannot be written as $P = \frac{P_0+P_1}{2}$ with $P_1 \neq P_2$ shift-invariant,

Convex analysis interpretation

- $\mathcal{I} = \{$shift-invariant measures$\}$ is a compact convex set,
- $\mathscr{E} = \{$shift-invariant ergodic measures$\}$ is the set of extremal points.

Shift-invariant measures

Ergodic shift-invariant measures have better constructive properties than non-ergodic ones.

Let P be a computable shift-invariant measure.

Birkhoff's ergodic theorem: speed of convergence

- P ergodic: computable speed [Avigad, Gerhardy, Towsner 2010]
- P non-ergodic: not computable in general [V'yugin 1997]

Birkhoff's ergodic theorem: randomness

- P ergodic: holds at Schnorr P-random sequences [Gács, H., Rojas 2009 - Avigad, Gerhardy, Towsner 2010]
- P non-ergodic: holds at Martin-Löf P-random sequences [V'yugin 1997 - Franklin, Towsner 2012]

Shift-invariant measures

Ergodic shift-invariant measures have better constructive properties than non-ergodic ones.

Let P be a computable shift-invariant measure.

Birkhoff's ergodic theorem: randomness

- P ergodic: if C is a Π^0_1-class then density($\{n : \sigma^n(x) \in C\}$) converge for every $x \in \mathrm{ML}_P$

  [Bienvenu, Day, H., Mezhirov, Shen 2010 - Franklin, Greenberg, Miller, Ng 2010]

- P non-ergodic: ?

Effective dimension and randomness

- P ergodic: $\dim(x) = \mathrm{Dim}(x)$ for every $x \in \mathrm{ML}_P$

  [Hochman 2009 - H. 2012]

- P non-ergodic: ?

Shift-invariant measures

Let P be a computable shift-invariant measure. TFAE:

- the speed of convergence is computable,
- the limiting frequencies along a sequence $x \in \mathsf{ML}_P$ with deficiency $\leq d$ are computable in (x, d),
- the ergodic decomposition of P is computable.

These conditions are not always satisfied.

- V'yugin's counter-example is a countably infinite combination of ergodic measures
- is it possible to construct a finite combination?

Shift-invariant measures

Let $P \neq Q$ be ergodic measures such that $\frac{P+Q}{2}$ is computable.
Must P and Q be computable?

The function

$$\begin{array}{rccc} h: & \mathscr{E} \times \mathscr{E} & \to & \mathcal{I} \\ & (P, Q) & \mapsto & \frac{P+Q}{2} \end{array}$$

is computable and one-to-one.

1. Is h^{-1} computable?
   No: h^{-1} is not even continuous!
2. Is h^{-1} computably invariant? i.e., $h(x)$ computable $\Rightarrow x$ computable?

h^{-1} is not continuous

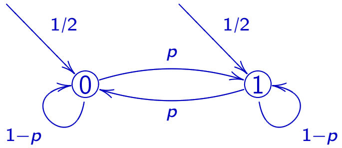


Figure: μ_p ergodic ($p > 0$)

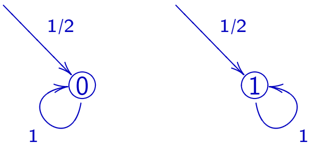


Figure: $\mu_0 = \frac{\delta_{\overline{0}}+\delta_{\overline{1}}}{2}$ non-ergodic

$$\text{When } p \to 0, \quad \underbrace{h(\mu_p, \mu_p)}_{\mu_p} \to \underbrace{h(\delta_{\overline{0}}, \delta_{\overline{1}})}_{\mu_0}$$

$$\text{but} \quad (\mu_p, \mu_p) \not\to (\delta_{\overline{0}}, \delta_{\overline{1}}).$$

Shift-invariant measures

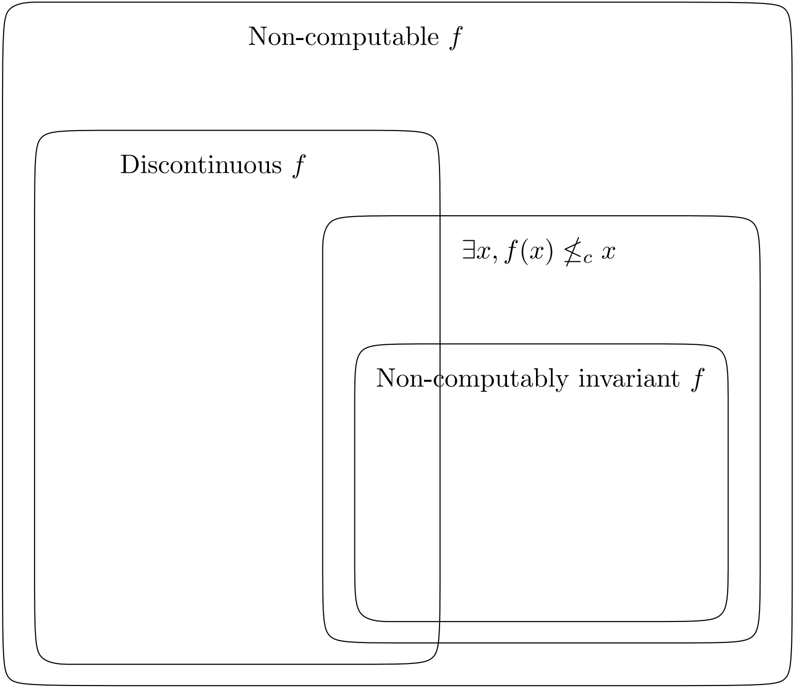
Let $P \neq Q$ be ergodic measures such that $\frac{P+Q}{2}$ is computable.
Must P and Q be computable?

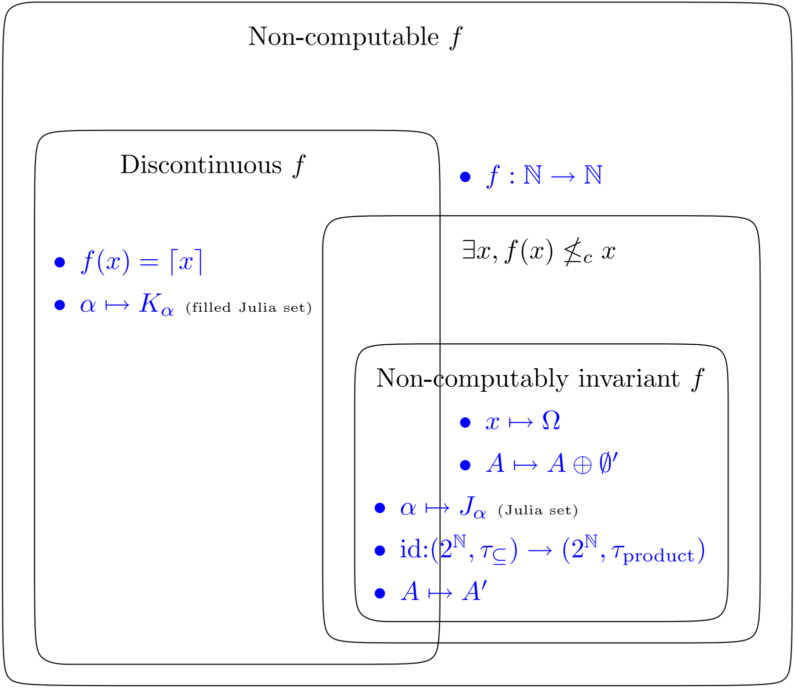
The function

$$\begin{array}{rccc} h: & \mathscr{E} \times \mathscr{E} & \to & \mathcal{I} \\ & (P, Q) & \mapsto & \frac{P+Q}{2} \end{array}$$

is computable and one-to-one.

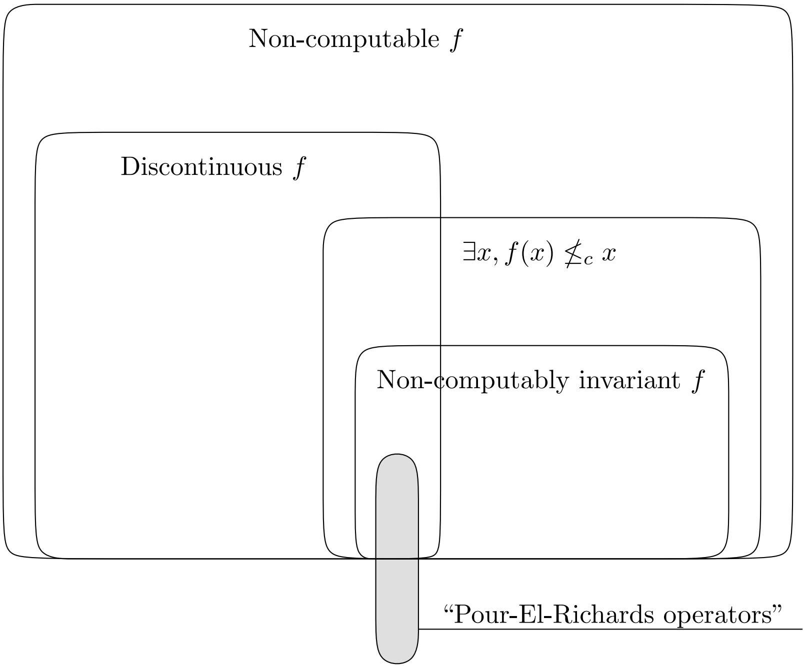
1. Is h^{-1} computable?
   No: h^{-1} is not even continuous!
2. Is h^{-1} computably invariant? i.e., $h(x)$ computable $\Rightarrow$ x computable?
   This talk: No. And more...


Non-computable f
Discontinuous f
$\exists x, f(x) \not\leq_c x$
Non-computably invariant f


Non-computable f
Discontinuous f
• $f(x) = \lceil x \rceil$
• $\alpha \mapsto K_\alpha$ (filled Julia set)
• $f : \mathbb{N} \to \mathbb{N}$
$\exists x, f(x) \not\leq_c x$
Non-computably invariant f
• $x \mapsto \Omega$
• $A \mapsto A \oplus \emptyset'$
• $\alpha \mapsto J_\alpha$ (Julia set)
• id:$(2^{\mathbb{N}}, \tau_{\subseteq}) \to (2^{\mathbb{N}}, \tau_{\text{product}})$
• $A \mapsto A'$

"Functions considered in analysis that are not computable are often not computable simply because they are discontinuous. [...] a noncomputable function may nevertheless be computably invariant; [...] and if it is not computably invariant, it can be a challenge to construct a computable element of the domain that is mapped to a noncomputable element. Pour-El and Richards [80] have shown a general result that shows that for linear operators the situation is simpler."

(V. Brattka, P. Hertling, and K. Weihrauch. A tutorial on computable analysis. In New Computational Paradigms, pages 425–491. Springer, 2008.)


Non-computable f
Discontinuous f
$\exists x, f(x) \not\leq_c x$
Non-computably invariant f
“Pour-El-Richards operators”

Theorem (Pour-El, Richards)

Let X and Y be effective Banach spaces and $T : X \to Y$ a closed linear operator whose domain contains a dense computable sequence e_n such that $T(e_n)$ is a computable sequence. The following are equivalent:

- *T is bounded (i.e., continuous),*
- *T is computably invariant,*
- *T is computable.*

Examples

The following operators are unbounded

- $\mathrm{id} : L^1[0,1] \to L^2[0,1]$,
- $\mathrm{id} : L^1[0,1] \to \mathscr{C}[0,1]$,
- $\frac{d}{dx} : \mathscr{C}[0,1] \to \mathscr{C}[0,1]$,
- solution operator of the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ from $\mathscr{C}(D)$ to $\mathscr{C}(D')$.

reminder: $\mathscr{C}[0,1] \subseteq L^2[0,1] \subseteq L^1[0,1]$

Following the line of Pour-El and Richards, find strong forms of discontinuities that prevent to be computably invariant

$$\text{discontinuous} \implies \text{non-computable}$$
$$\underbrace{\text{strongly}}_{?}\ \text{discontinuous} \implies \text{non-computably invariant}$$

Several objectives:

- better understand the interaction between topology and computability,
- reduce a computability-theoretic property to a topological property, usually simpler to verify,
- have one single abstract construction that can be applied easily in many situations.

Constructions in computability theory

Derivative

- Myhill, 1971: a computable function $f \in \mathscr{C}^1[0,1]$ such that f' is not computable,
- Pour-El, Richards, 1989: the linear operator $f \mapsto f'$ is unbounded

Differentiability

- Bolzano, 1830 - Riemann, 1861: a computable, nowhere differentiable function $f(x) = \sum_n \frac{\sin(n^2 \pi x)}{n^2}$,
- Banach, 1931: the set of nowhere differentiable continuous functions is co-meager + computable Baire theorem

Tuing incomparability

- Kleene-Post, 1954: $A, B <_T \emptyset'$ such that $A|_T B$,
- Jockush, 1977: the set of $A \oplus B$ s.t. $A|_T B$ is co-meager + $\emptyset'$-computable Baire theorem

Effective topology

- An ***effective topological space*** if a set X with a topology with a countable basis $B_1, B_2, B_3, \ldots$, with computable intersection.
- A point $x \in X$ is ***computable*** if $N(x) := \{n : x \in B_n\}$ is c.e.
- An open set $U \subseteq X$ is ***effective*** if there is a c.e. set $W \subseteq \mathbb{N}$ such that $U = \bigcup_{n \in W} B_n$.
- A function $f : X \to Y$ is ***computable*** if there is a machine M that on any enumeration of $N(x)$ enumerates $N(f(x))$.
- Equivalently, f is computable iff f is ***effectively continuous***, i.e. $f^{-1}(B_n)$ is an effective open set, uniformly in n.
- $f : X \to Y$ is ***computably invariant*** if for every computable x, $f(x)$ is computable.

Effective topology

Examples

- Cantor space $(2^{\mathbb{N}}, \tau_{\text{product}})$
  - basis: cylinders $[0], [1], [00], [01], [10], \ldots$
  - computable elements: the computable sequences/sets/reals
- $(2^{\mathbb{N}}, \tau_{\subseteq})$
  - basis: Scott open sets $\{E \subseteq \mathbb{N} : F \subseteq E\}$ with F finite
  - computable elements: the c.e. sets
- real interval $([0, 1], \tau_{\text{Euclidean}})$
  - basis: (a, b) with $a, b \in \mathbb{Q}$
  - computable elements: the computable real numbers
- real interval $([0, 1], \tau_{\leq})$
  - basis: $(a, 1]$ with $a \in \mathbb{Q}$
  - computable elements: the left-c.e. real numbers
- effective Polish space
  - basis: metric balls $B(s, q)$ with s in a countable dense subset and $q \in \mathbb{Q}$
  - computable elements: the limits of fast computable Cauchy sequences

Let $h : X \to Y$ be continuous and one-to-one.

Definition

- h is ***continuously invertible at*** x if h^{-1} is continuous at $h(x)$.
- h is ***locally continuously invertible at*** x if there is a neighborhood B of x such that $h_{|B} : B \to h(B)$ is continuously invertible at x.

Game interpretation

player: describes $h(x)$

opponent: tries to guess x.

- h is continuously invertible at x if the opponent wins.
- h is locally continuously invertible at x if the opponent wins *with the help of an advice* $x \in B$.

If X is compact and Y is Hausdorff then h^{-1} is continuous.

Counter-examples arise when

- X is not compact
- or Y is not Hausdorff.

Examples

Non-Hausdorff Y

1. Enumeration

$$\mathrm{id} : (2^{\mathbb{N}}, \tau_{\text{product}}) \to (2^{\mathbb{N}}, \tau_{\subseteq})$$

player: enumerates $A \subseteq \mathbb{N}$.

opponent: tries to describe the characteristic function of A.

- continuously invertible: at $\mathbb{N}$,
- locally continuously invertible: at every co-finite set.

Examples

Non-Hausdorff Y

2. Approximation from below

$$\mathrm{id} : (2^{\mathbb{N}}, \tau_{\text{product}}) \to ([0, 1], \tau_{\leq})$$

player: approximates $\overline{0.A}$ from below.

opponent: tries to describe the characteristic function of A.

- continuously invertible: at $\mathbb{N}$,
- locally continuously invertible: at every co-finite set.

Examples

Compact X

3. Unique choice in Cantor space

$$\begin{array}{rcl} (2^{\mathbb{N}}, \tau_{\text{product}}) & \to & (\text{singletons}(2^{\mathbb{N}}), \tau_{\text{upper Vietoris}}) \\ x & \mapsto & \{x\} \end{array}$$

player: enumerates $2^{\mathbb{N}} \setminus \{x\}$.

opponent: tries to describe x.

- continuously invertible: everywhere.
- locally continuously invertible: everywhere.

Examples

Non-compact X

4. Unique choice in $\mathbb{N}$

$$\begin{array}{rcl} \mathbb{N} & \to & (\text{closed sets}(\mathbb{N}), \tau_{\text{upper Vietoris}}) \\ n & \mapsto & \{n\} \end{array}$$

player: enumerates $\mathbb{N} \setminus \{n\}$.

opponent: tries to find n.

- continuously invertible: nowhere
- locally continuously invertible: everywhere.

Examples

Non-compact X

5. Pseudo-unique choice in Baire space

$$\mathrm{id} : (\mathbb{N}^{\mathbb{N}}, \tau_{\mathrm{product}}) \to (\mathbb{N}^{\mathbb{N}}, \tau_{\mathrm{neg}})$$

player: enumerates the complement of Graph(f).

opponent: tries to describe f.

- continuously invertible: nowhere.
- locally continuously invertible: nowhere.

Examples

Non-compact X

6. Ergodic decomposition

$$\begin{array}{rccc} h: & \mathscr{E} \times \mathscr{E} & \to & \mathcal{I} \\ & (P, Q) & \mapsto & \frac{P+Q}{2} \end{array}$$

player: describes $\frac{P+Q}{2}$.
opponent: tries to describe P and Q.

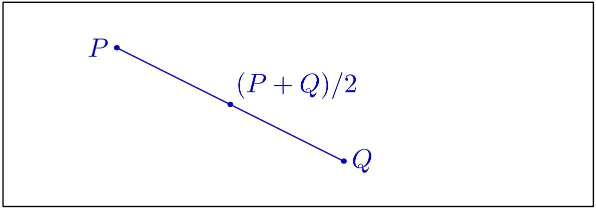
- continuously invertible: on the diagonal.
- locally continuously invertible: on the diagonal.

Examples

Proposition

Let $P \neq Q$ be ergodic. h is not locally continuously invertible at (P, Q).

Proof.



Start with $P \neq Q$ are ergodic.

Examples

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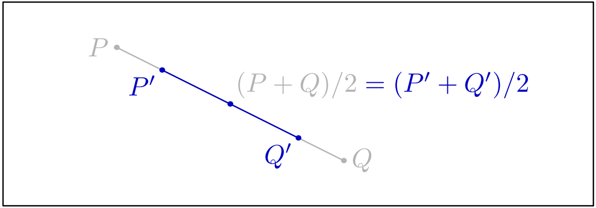

Let $0 < \lambda < 1$ and

$$P' := \lambda P + (1 - \lambda) Q,$$
$$Q' := (1 - \lambda) P + \lambda Q.$$

$\frac{P' + Q'}{2} = \frac{P + Q}{2}$ but P' and Q' are not ergodic!

Examples

Proposition

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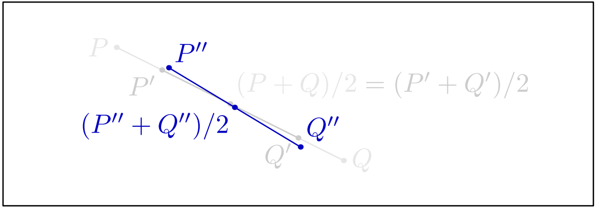

Take P'', Q'' ergodic such that

$$P'' \approx P',$$
$$Q'' \approx Q'.$$

Hence $\frac{P''+Q''}{2} \approx \frac{P+Q}{2}$. □

Examples

Non-compact X

7. Linear operator

Let T be linear and one-to-one.

player: describes $T(x)$.

opponent: tries to describe x.

- If T^{-1} is bounded, T is continuously invertible everywhere.
- If T^{-1} is unbounded, T is nowhere locally continuously invertible.
  - $\mathrm{id} : \mathscr{C}[0,1] \to L^1[0,1]$,
  - $\mathrm{id} : L^2[0,1] \to L^1[0,1]$,
  - $I : \mathscr{C}[0,1] \to \mathscr{C}[0,1]$, $I(f) : x \mapsto \int_0^x f(t)\mathrm{d}t$.

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reminder: $\mathscr{C}[0,1] \subseteq L^2[0,1] \subseteq L^1[0,1]$

Theorem

Let h be computable one-to-one and x be 1-generic.

- *Either h is locally continuously invertible at x,*
- *or $x \not\leq_c h(x)$.*

Proof

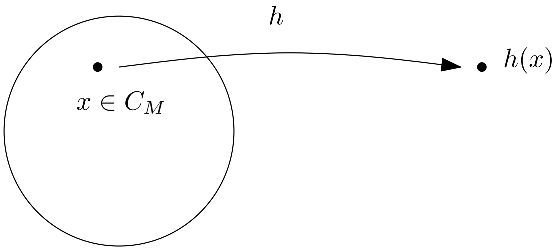
Let M be a machine and

$$C_M = \{x : M^{h(x)} = x\}$$
$$U_M = \{x : M^{h(x)} \downarrow \neq x\}.$$

Lemma

If h is not locally continuously invertible at $x \in C_M$ then $x \in \overline{U_M}$.

Proof.



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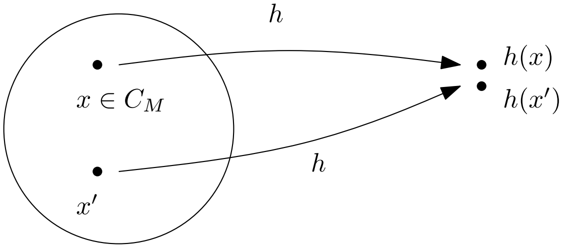

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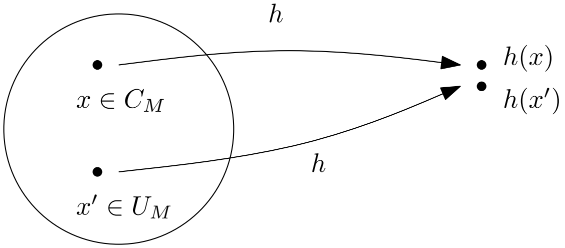


□

Direct applications

- If $A \in 2^{\mathbb{N}}$ is 1-generic then it cannot be computed from its enumeration.
- If (P, Q) is 1-generic then $(P, Q) \not\leq_c \frac{P+Q}{2}$.
- If T^{-1} is unbounded and x is 1-generic then $x \not\leq_c T(x)$.

We want more:

- what makes h not locally continuous invertible at the 1-generic points?
- find x non-computable such that $h(x)$ is computable.

A weak form of continuity for h^{-1}

h^{-1} is continuous $\iff$ h is open.

Definition

h is ***quasi-open*** if for every (non-empty) open set U, $h(U)$ has non-empty interior (in the subspace $h(X)$).
h is ***somewhere quasi-open*** if there is a (non-empty) open set B such that $h_{|B} : B \to h(B)$ is quasi-open.

Formally, h is quasi-open if

$$\forall U \subseteq X, \exists V \subseteq Y \text{ such that } \emptyset \neq h^{-1}(V) \subseteq U.$$

and h is somewhere quasi-open if

$$\exists B \subseteq X, \forall U \subseteq B, \exists V \subseteq Y \text{ such that } \emptyset \neq h^{-1}(V) \cap B \subseteq U.$$

A strong form of discontinuity for h^{-1}

Definition

h is ***nowhere quasi-open*** if h is not somewhere quasi-open.

- h is nowhere quasi-open means that h^{-1} badly fails to be continuous.
- h is nowhere quasi-open if for every B there exists $U_B \subseteq B$ such that $h(B \setminus U_B)$ is dense in $h(B)$.

Observation

- $h_{|B} : B \to h(B)$ is not continuously invertible at any $x \in U_B$.

Game interpretation

- player: describes $h(x)$
- opponent: tries to guess x.

The player can defeat the opponent: start with any $x \in U_B$; if the opponent eventually guesses that $x \in U_B$, move x to $x' \in B \setminus U_B$.

Examples

1. Enumeration

$$\mathrm{id}_{\subseteq} : (2^{\mathbb{N}}, \tau_{\mathrm{product}}) \to (2^{\mathbb{N}}, \tau_{\subseteq})$$

is nowhere quasi-open.

- To $B = [w]$ associate $U_B = [w0]$.
- From any $A \in [w0]$ the player can move to $[w1]$ coherently with the enumeration of A.

2. Approximation from below

$$\mathrm{id}_{\leq} : ([0,1], \tau_{\mathrm{Euclidean}}) \to ([0,1], \tau_{\leq})$$

is nowhere quasi-open.

- To $B = (a, b)$ associate $U_B = (a, \frac{a+b}{2})$.
- From any $x \in (a, \frac{a+b}{2})$ the player can move to $(\frac{a+b}{2}, b)$ coherently with the approximation of x from below.

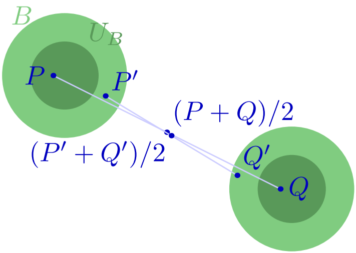
Examples

3. Ergodic decomposition

$$h: \begin{array}{ccc} \mathscr{E} \times \mathscr{E} & \to & \mathcal{I} \\ (P, Q) & \mapsto & \frac{P+Q}{2} \end{array}$$

is nowhere quasi-open.

- To B associate some $U_B \subseteq B$ far from the diagonal $\{(P, P) : P \in \mathscr{E}\}$.
- From any $(P, Q) \in U$ the player can move to $(P', Q') \in B \setminus U_B$ such that $\frac{P'+Q'}{2} \approx \frac{P+Q}{2}$.



Examples

4. Linear operator

Proposition

Let T be computable linear one-to-one such that T^{-1} is unbounded. T is nowhere quasi-open.

- To a ball $B(s, r)$ associate $U_B = B(s, \frac{r}{3})$.
- Take a such that $\|a\| < \frac{r}{3}$ and $\|T(a)\| < \epsilon$. From $x \in B(s, \frac{r}{3})$ the player can move to $x + a \in B(s, r) \setminus B(s, \frac{s}{3})$, with $T(x + a) \approx_\epsilon T(x)$.

Examples

5. Pseudo-unique choice in Baire space

$$\mathrm{id} : (\mathbb{N}^{\mathbb{N}}, \tau_{\text{product}}) \to (\mathbb{N}^{\mathbb{N}}, \tau_{\text{neg}})$$

is nowhere quasi-open.

- To $B = [(n_1, v_1), \ldots, (n_k, v_k)]$ associate $U_B = [(n_1, v_1), \ldots, (n_k, v_k), (n, 0)]$, fresh n.
- The player can move from $f \in U_B$ to f' mapping n to some fresh $v \neq 0$.

B and U_B can be assumed w.l.o.g. to be basic open sets.

Definition

h is ***effectively nowhere quasi-open*** if U_B can be computed from B.

Theorem A

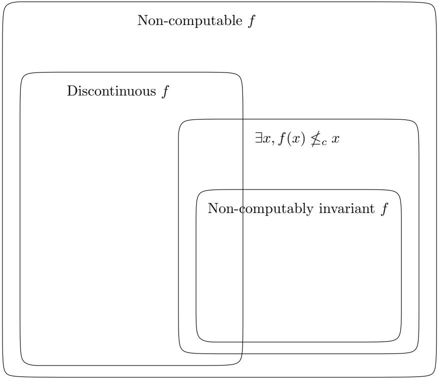
Let h be computable and effectively nowhere quasi-open. If x is 1-generic then

- *h is not locally continuously invertible at x,*
- *hence $x \not\leq_c h(x)$.*

Let X be an effective Polish space and Y an effective topological space. Let $h : X \to Y$ be computable one-to-one and effectively nowhere quasi-open.

Theorem A

For each 1-generic x, $x \not\leq_c h(x)$.



Examples

If $h_{|C} : C \to h(C)$ is an homeomorphism then C is nowhere dense.

1. Enumeration

$$\mathrm{id} : (2^{\mathbb{N}}, \tau_{\mathrm{product}}) \to (2^{\mathbb{N}}, \tau_{\subseteq})$$

- let $C = (01 + 10)^*$. $\mathrm{id} : (C, \tau_{\subseteq}) \to (C, \tau_{\mathrm{product}})$ is computable.

2. Approximation from below

$$\mathrm{id} : ([0, 1], \tau_{\mathrm{Euclidean}}) \to ([0, 1], \tau_{\leq})$$

- if $\mathrm{id} : (C, \tau_{\leq}) \to (C, \tau_{\mathrm{Euclidean}})$ is continuous then $C = \emptyset$ or $C = \{x\}$ for some x.

Examples

3. Ergodic decomposition

$$\begin{array}{rccc} h: & \mathscr{E} \times \mathscr{E} & \to & \mathfrak{I} \\ & (P, Q) & \mapsto & \frac{P+Q}{2} \end{array}$$

- let $C = \text{Bernoulli} \times \text{Bernoulli}$. $h^{-1}_{|h(C)}$ is computable.

4. Integration

$$\begin{array}{rccc} I: & \mathscr{C}[0,1] & \to & \mathscr{C}[0,1] \\ & f & \mapsto & (x \mapsto \int_0^x f(t)\mathrm{d}t) \end{array}$$

- let $C = \text{1-Lipschitz}$. $\frac{d}{dx} : I(C) \to C$ is computable.

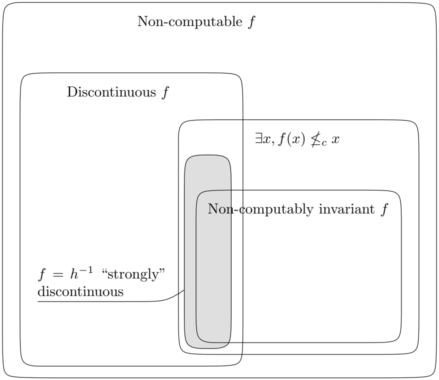
Let X be an effective Polish space and Y an effective topological space. Let $h : X \to Y$ be computable one-to-one and effectively nowhere quasi-open.

Theorem A

For each 1-generic x, $x \not\leq_c h(x)$.

Theorem B

h^{-1} is not computably invariant, i.e., there exists a non-computable x such that $h(x)$ is computable.



Theorem B

h^{-1} is not computably invariant, i.e., there exists a non-computable x such that $h(x)$ is computable.

Proof.

Finite injury argument.
$R_e : M_e$ does not compute x.

- x is restrained from leaving a ball B by higher priority requirements.
- R_e restrains x from leaving U_B.
- R_e tests whether M_e computes a point inside U_B.
- If the test succeeds at some stage, move x to $B \setminus U_B$, remaining coherent with the current description of $h(x)$. □

Well-known examples

1. Enumeration

$$\mathrm{id} : (2^{\mathbb{N}}, \tau_{\text{product}}) \to (2^{\mathbb{N}}, \tau_{\subseteq})$$

Thm A If $A \subseteq \mathbb{N}$ is 1-generic then it cannot be computed from the enumeration of A.

Thm B There exists a non-computable c.e. set [Turing, 1936].

Well-known examples

2. Pseudo-unique choice in Baire space

$$\mathrm{id}: \quad (\mathbb{N}^{\mathbb{N}}, \tau_{\text{product}}) \quad \to \quad (\mathbb{N}^{\mathbb{N}}, \tau_{\text{neg}})$$

is effectively nowhere quasi-open.

Thm A If f is 1-generic then it cannot be computed the co-enumeration of its graph.

Thm B There exists a non-computable $f : \mathbb{N} \to \mathbb{N}$ whose graph is co-c.e. In particular $\{f\}$ is Π^0_1.

Explicit construction:

$$f(n) = \begin{cases} 0 & \text{if } \varphi_n(n)\uparrow \\ t+1 & \text{if } \varphi_n(n)\downarrow_t \end{cases}$$

Well-known examples

3. Linear operators

- T computable linear one-to-one, T^{-1} unbounded

Thm A If x is 1-generic then $x \not\leq_c T(x)$.

Thm B There exists a non-computable x such that $T(x)$ is computable [Pour-El, Richards, 1989].

- $\int : \mathscr{C}[0,1] \to \mathscr{C}[0,1]$.

Thm A If $f \in \mathscr{C}[0,1]$ is 1-generic then $f \not\leq_c F$ where $F' = f$, $F(0) = 0$.

Thm B There exists a computable F such that F' is continuous but not computable [Myhill, 1971].

Examples

4. Ergodic decomposition

Thm A If (P, Q) is 1-generic then $(P, Q) \not\leq_c \frac{P+Q}{2}$,

Thm B There exist noncomputable ergodic P and Q such that $\frac{P+Q}{2}$ is computable.

Let X be an effective Polish space and Y an effective topological space. Let $h : X \to Y$ be computable, one-to-one and effectively nowhere quasi-open.

Theorem A

For each 1-generic x, $x \not\leq_c h(x)$.

Theorem B

h^{-1} is not computably invariant, i.e., there exists a non-computable x such that $h(x)$ is computable.

One can even prove

Theorem B′

$$J \leq_W h^{-1}.$$

Question

Is there x, generic in some sense, such that $h(x)$ is computable?

h-genericity

Let $h : X \to Y$ be computable one-to-one and effectively nowhere quasi-open.

We consider the case when X is compact and Y is not Hausdorff.

h-genericity

We assume that $h^{-1}(V)$ are computable clopen sets.
Let

$$\uparrow_h x := \bigcap_{V \ni h(x)} h^{-1}(V).$$

- $x' \in \uparrow_h x$ if the opponent can never reject x' when the player describes $h(x)$.
- $\uparrow_h x$ is Π^0_1 relative to $h(x)$.
- When Y is Hausdorff, $\uparrow_h x = \{x\}$.
- h is continuously invertible at $x \iff \uparrow_h x = \{x\}$.
- h is locally continuously invertible at $x \iff x$ is isolated in $\uparrow_h x$.

Dichotomy

- if h is locally continuously invertible at x then $x \leq_c h(x)$.
- if h is not locally continuously invertible at x and x is 1-generic then $x \not\leq_c h(x)$.

h-genericity

Definition

x is ***h*-generic** if x is 1-generic inside $\uparrow_h x$, i.e. for every effective open set U,

- either $x \in U$,
- or there exists a neighborhood B of x such that $B \cap U \cap \uparrow_h x = \emptyset$.

Observations

- 1-genericity implies h-genericity.
- If x isolated in $\uparrow_h x$ then x is (vacuously) h-generic.

Reminder

If h is not loc. cont. invertible at x and x is 1-generic then $x \not\leq_c h(x)$.

Actually

If h is not loc. cont. invertible at x and x is h-generic then $x \not\leq_c h(x)$.

h-genericity

Theorem C

There exists x such that

- *h is not locally continuously invertible at x,*
- *x is h-generic,*
- *$h(x)$ is computable.*

In particular, x is not computable.

- It can be combined with the finite extension method: given uniformly effective dense open sets U_n, x can be taken in $\bigcap_n U_n$.
- The construction makes x low.

Examples

1. Enumeration

$$\mathrm{id}_{\subseteq} : (2^{\mathbb{N}}, \tau_{\mathrm{product}}) \to (2^{\mathbb{N}}, \tau_{\subseteq}).$$

- $\uparrow_{\subseteq} A = \{B : A \subseteq B\}$.
- A is $\mathrm{id}_{\subseteq}$-generic if A belongs to every effective open set that is dense *above* A.
- Every co-finite set is $\mathrm{id}_{\subseteq}$-generic.
- Every co-infinite $\mathrm{id}_{\subseteq}$-generic set is
  - simple,
  - hypersimple,
  - of the form $A = A_0 \oplus A_1$ with $A_0 |_T A_1$,
  - not autoreducible.

Thm C There exists a co-infinite $\mathrm{id}_{\subseteq}$-generic c.e. set A.

Proposition

A co-infinite c.e. set A is $\mathrm{id}_{\subseteq}$-generic iff it is p-generic [Ingrassia, 1981].

Examples

2. Approximation from below

$$\mathrm{id}_{\leq} : (2^{\mathbb{N}}, \tau_{\mathrm{product}}) \to ([0,1], \tau_{\leq})$$

- $\uparrow_{\leq} A = \{B : A \leq B\} = [A, 1]$.
- A is $\mathrm{id}_{\leq}$-generic if A belongs to every effective open set that is dense *above* A.
- Every co-finite set is $\mathrm{id}_{\leq}$-generic.
- Every co-infinite $\mathrm{id}_{\leq}$-generic set is weakly 1-generic.

Thm C There exists a co-infinite left-c.e. A that is 1-generic inside $[x, 1]$.

$\equiv$ construction of a left-c.e. weakly 1-generic set.

Examples

3. Π^0_1-class

$\mathsf{CL}(2^{\mathbb{N}})$: non-empty closed subsets of $2^{\mathbb{N}}$.

$$\mathsf{id}_{\mathsf{miss}} : (\mathsf{CL}(2^{\mathbb{N}}), \tau_{\text{hit-or-miss}}) \to (\mathsf{CL}(2^{\mathbb{N}}), \tau_{\text{miss}})$$

- $\mathsf{id}_{\mathsf{miss}}$ is continuously invertible at every singleton.
- $\mathsf{id}_{\mathsf{miss}}$ is locally continuously invertible at every finite set.
- $\mathsf{id}_{\mathsf{miss}}$ is effectively nowhere quasi-open: given a finite prescription $\mathcal{B} = (u_1^+, \dots, u_n^+, v_1^-, \dots, v_k^-)$, let $\mathcal{U} = \mathcal{B} \cup (w0^+, w1^+)$ for some fresh w.
- $\uparrow_{\mathsf{miss}} C = \{C' : C' \subseteq C\}$.

Thm C There exists a non-empty $\mathsf{id}_{\mathsf{miss}}$-generic Π^0_1-class C without isolated point.

- Such a class has no computable member.

Back to randomness

There exist P, Q ergodic non-computable such that $\frac{P+Q}{2}$ is computable.

Question

$\mathrm{ML}_{\frac{P+Q}{2}} \subseteq \mathrm{ML}_P \cup \mathrm{ML}_Q$?

Answer

No.

- P and Q can be constructed above $\emptyset'$. Let $x \in \mathrm{ML}_{\frac{P+Q}{2}}$ be Δ^0_2: $x \notin \mathrm{ML}_P \cup \mathrm{ML}_Q$.
- It provides an ergodic measure P for which Hippocratic randomness is strictly weaker than uniform randomness [Monin].

Questions

- Is $\mathrm{ML}_{\frac{P+Q}{2}} \not\subseteq \mathrm{ML}_P \cup \mathrm{ML}_Q$ true for every sufficiently generic pair (P, Q)?
- Is there a good notion of h-genericity when X is not compact and Y is not Hausdorff?
- What happens when the function h is replaced by a relation $R(x, y)$?
- Let
$$C_h = \{x \in X : h(x) \text{ is computable}\}.$$
Is it possible to develop resource-bounded Baire category on C_h?
  - C_h should not be "effectively meager",
  - for every $x \in C_h$, $\{x\}$ should be "effectively meager",
  - the set of computable x's should be "effectively meager".

Thank you!

Appendix: genericity

A point is "generic" if it avoids every "effective" small set, in the sense of Baire category.

A set C is ***nowhere dense*** if $\text{interior}(\overline{C}) = \emptyset$, or equivalently

1. $C \cap U = \emptyset$ for some dense open set U,
2. $C \subseteq \text{boundary}(U) = \overline{U} \setminus U$ for some open set U.

Each one has its effective version:

1. x is ***weakly*** 1***-generic*** if $x \in U$ for every effective dense open set U,
2. x is 1***-generic*** if $x \notin \text{boundary}(U)$ for every effective open set U.