Algorithmic Randomness and Pathological Computable Measures

Christopher P. Porter

University of Notre Dame

Joint Mathematics Meetings
Special Session on the Life and Legacy of Alan Turing
January 4, 2012

Motivation

What is an algorithmically random sequence?

Intuitively, a sequence is algorithmically random if it contains no "effectively specifiable regularities".

In the absence of such regularities, algorithmically random sequences are not detected as non-random by some effective test for randomness.

In other words, if a sequence contains some "effectively specifiable regularity", there is some effective test for randomness that detects the sequence as non-random.

Towards a Formal Definition of Algorithmic Randomness

There are a number of ways one can formally characterize the algorithmic randomness:

- in terms of effective unpredictability;
- in terms of effective incompressibility;
- in terms of effective typicality.

Towards a Formal Definition of Algorithmic Randomness

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- in terms of effective incompressibility;
- in terms of effective typicality. $\Leftarrow$

Today, we'll focus on this third way of characterizing randomness.

Fixing Some Notation

$2^{<\omega}$ is the collection of finite binary sequences.

2^{ω} is the collection of infinite binary sequences.

The standard topology on 2^{ω} is given by the basic open sets

$$[\sigma] = \{X \in 2^{\omega} : \sigma \prec X\}$$

where $\sigma \in 2^{<\omega}$ and $\sigma \prec X$ means that σ is an initial segment of X.

Lastly, the Lebesgue measure on 2^{ω}, denoted λ, is defined by

$$\lambda([\sigma]) = 2^{-|\sigma|}$$

for each $\sigma \in 2^{<\omega}$ (where $|\sigma|$ is the length of σ), and then we extend λ to all Borel sets in the usual way.

Martin-Löf Randomness

Definition

A *Martin-Löf test* is a uniform sequence $(\mathcal{U}_i)_{i\in\omega}$ of Σ^0_1 (i.e. effectively open) subsets of 2^ω such that for each i,

$$\lambda(\mathcal{U}_i) \leq 2^{-i}.$$

A sequence $X \in 2^\omega$ *passes the Martin-Löf test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.

$X \in 2^\omega$ is *Martin-Löf random*, denoted $X \in \mathrm{MLR}$, if X passes *every* Martin-Löf test.

Schnorr Randomness

Definition

A *Schnorr test* is a Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ such that for each i,

$$\lambda(\mathcal{U}_i) = 2^{-i}.$$

A sequence $X \in 2^\omega$ *passes the Schnorr test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.

$X \in 2^\omega$ is *Schnorr random*, denoted $X \in \mathrm{SR}$, if X passes *every* Schnorr test.

Fact: $\mathrm{MLR} \subsetneq \mathrm{SR}$.

Computable Probability Measures on 2^ω

Definition

A probability measure μ on 2^ω is *computable* if $\sigma \mapsto \mu([\sigma])$ is computable as a real-valued function.

In other words, μ is computable if there is a computable function $\hat{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that

$$|\mu([\sigma]) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

We've already seen one example of a computable measure: the Lebesgue measure.

Measures on 2^ω other than the Lebesgue measure are *non-uniform measures.*

Randomness with respect to non-uniform measures

We can also define Martin-Löf randomness and Schnorr randomness with respect to a non-uniform computable measure μ:

μ-Martin-Löf tests:	$\mu(\mathcal{U}_i) \leq 2^{-i}$
μ-Schnorr tests:	$\mu(\mathcal{U}_i) = 2^{-i}$

Let MLR_μ denote the collection of μ-Martin-Löf random sequences.

Let SR_μ denote the collection of μ-Schnorr random sequences.

In general, we have $\mathsf{MLR}_\mu \subseteq \mathsf{SR}_\mu$.

Let $\mathsf{MLR}_{comp} = \{X \in 2^\omega : X \in \mathsf{MLR}_\mu \text{ for some computable } \mu\}$.

Pathological Computable Measures: Atomic Measures

Definition

μ is atomic if there is some $X \in 2^\omega$ such that $\mu(\{X\}) > 0$.
In this case, we say that X is a μ-atom.

Let Atom_μ denote the collection of μ-atoms.

Note that $\mathrm{Atom}_\mu \subseteq \mathrm{MLR}_\mu$.

Further, $X \in \mathrm{Atom}_\mu$ for some computable measure μ implies that X is computable.

Lastly, X is not random with respect to any continuous computable measure, denoted $X \in \mathrm{NCR}_{comp}$, if for every non-atomic (thus continuous) μ, we have $X \notin \mathrm{MLR}_\mu$.

Pathological Computable Measures: Trivial Measures

The most pathological case is the one in which the support of the measure μ consists *entirely* of μ-atoms.

Definition

μ is trivial if $\mu(\mathsf{Atom}_\mu) = 1$.

Clear example: $\mu(\{0^\infty\}) = 1$.

However, not all trivial measures are *this* trivial: a number of different pathologies can occur.

Pathology 1

Proposition (Porter, Bienvenu)

There exists a trivial computable measure μ such that

$$\mathrm{MLR}_\mu \neq \mathrm{Atom}_\mu.$$

In fact, we can cook up measures μ so that

$$\mathrm{MLR}_\mu = \mathrm{Atom}_\mu \cup \{X_1, X_2, \ldots, X_n\}$$

for any $n \in \omega$.

Moreover, each X_i *must* be a member of NCR_{comp}.

Pathology 2

Claim (Schnorr)

$\mathrm{MLR}_\mu = \mathrm{SR}_\mu$ if and only if μ is trivial.

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$\mathrm{MLR}_\mu = \mathrm{SR}_\mu$ if and only if μ is trivial.

Theorem (Porter)

There is a trivial computable measure μ such that

$$\mathrm{SR}_\mu \neq \mathrm{Atom}_\mu$$

and

$$\mathrm{MLR}_\mu = \mathrm{Atom}_\mu.$$

LR-degrees

Definition

For $A, B \in 2^\omega$, A is LR-reducible to B, denoted $A \leq_{LR} B$ if and only if

$$\mathrm{MLR}^B \subseteq \mathrm{MLR}^A.$$

Moreover, A and B are LR-equivalent, denoted $A \equiv_{LR} B$, if and only if $A \leq_{LR} B$ and $B \leq_{LR} A$.

The LR-degree of $A \in 2^\omega$ is $\{X \in 2^\omega : A \equiv_{LR} X\}$.

Let $\mathscr{D}_{LR}$ denote the collection of LR-degrees.

Note: $\mathscr{D}_{LR}$ is *uncountable*.

$LR(\mu)$-degrees

We can also consider the $LR(\mu)$-degrees associated to a computable measure μ, denoted $\mathscr{D}_{LR(\mu)}$.

- In the case that $\mathrm{MLR}_\mu = \mathrm{Atom}_\mu$, there is only one $LR(\mu)$-degree.
- If $\mathrm{MLR}_\mu = \mathrm{Atom}_\mu \cup \{X\}$ for some $X \notin \mathrm{Atom}_\mu$, there are exactly two $LR(\mu)$-degrees.
- With some care, we can get any finite number of degrees.

Pathology 3

Theorem (Porter)

For every finite distributive lattice $(\mathscr{L}, \leq)$, there is a computable trivial measure μ such that

$$(\mathscr{L}, \leq) \cong (\mathscr{D}_{LR(\mu)}, \leq_{LR(\mu)}).$$

For each such measure μ in the proof, $\mathrm{MLR}_\mu \subseteq \mathrm{NCR}_{comp}$.

Locating the Pathologies

Despite this pathological behavior, we can classify the Turing degrees in which such pathologies occur:

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Theorem (Porter, Bienvenu)

Given a Turing degree **a** *containing some* $A \in \mathrm{MLR}$, *there is some* $B \in \mathbf{a}$ *such that*

$$B \in \mathrm{MLR}_{comp} \cap \mathrm{NCR}_{comp}$$

if and only if **a** *is hyperimmune.*