E-Closed Structures

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Gerald E. Sacks
Harvard University
MIT

Normann Schemes

$\overrightarrow{x}$ is $x_1, ... x_n$

(1) projection	$\{e\}(\overrightarrow{x}) = x_i$	if $e =< 1, n, i >$
(2) difference	$\{e\}(\overrightarrow{x}) = x_i - x_j$	if $e =< 2, n, i, j >$.
(3.1) pairing	$\{e\}(\overrightarrow{x}) = \{x_i, x_j\}$	if $e =< 3, 1, n, i, j >$.
(3.2) union	$\{e\}(\overrightarrow{x}) = \cup\{y \mid y \in x_1\}$	if $e =< 3, 2, n >$.

More Normann Schemes

(4) E-recursive bounding

$$\{e\}(\overrightarrow{x}) = \{\{c\}(y) \mid y \in x_1\}$$

$$\text{if } e =< 4, n, c >.$$

(5) composition

$$\{e\}(\overrightarrow{x}) = \{c\}(\{d_1\}(\overrightarrow{x}), \ldots, \{d_m\}(\overrightarrow{x}))$$

$$\text{if } e =< 5, n, m, c, d_1, \ldots, d_m >$$

(6) enumeration

$$\{e\}(c, \overrightarrow{x}, \overrightarrow{y}) = \{c\}(\overrightarrow{x})$$

$$\text{if } e =< 6, m, n >.$$

E-Recursion Theory

E is the **class of E-recursive evaluations**.

A tuple $< e, \overrightarrow{x}, y >$ is put in E

iff the schemes determine a value y for $\{e\}(\overrightarrow{x})$.

E is the range of a Σ_1 transfinite recursion on ORD.

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f is a **partial E-recursive** function

iff $\exists e \, \forall \overrightarrow{x} \quad f(\overrightarrow{x}) = \{e\}(\overrightarrow{x})$.

E-Recursive Enumerability

A is **E-recursively enumerable in y** iff

$$\exists e\ A = \{x \mid \{e\}(x, y) \downarrow\}.$$

A is **E-recursively enumerable** iff A is E-RE in $\emptyset$.

There exists a total E-recursive function

whose range is not E-RE in any y.

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If A and its complement are E-RE, then A is E-recursive.

Computations

A **computation instruction** is an $(n+1)-$tuple $< e, \overrightarrow{x} >$.

$< e, \overrightarrow{x} >$ is the top node of the computation tree $T_{<e, \overrightarrow{x}>}$.

Every other node is an **immediate subcomputation instruction** of the node above it.

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If $e =< 5, 1, 1, c, d >$,
then $< d, x >$ is an im. sub. in. of $< e, \overrightarrow{x} >$.

If $\{d\}(x) = y$, then $< c, y >$ is an im. sub. in. of $< e, \overrightarrow{x} >$.

More Computations

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- The relation $b >_U a$ is E-RE but not E-REC.
- The relation $\exists c \; b >_U c >_U a$ is not E-RE.
- **Lemma** $\{e\}(\overrightarrow{x}) \downarrow \longleftrightarrow T_{<e, \overrightarrow{x}>}$ is wellfounded.

Divergence Witnesses

Suppose $\{e\}(x)\uparrow$ (diverges). Then $T_{<e,x>}$ is illfounded.

A **witness w to the divergence of $\{\mathbf{e}\}(\mathbf{x})$** is any ∞ branch of $T_{<e,x>}$.

- w is a function with domain ω.

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- w is a function with domain ω.
- $w(0) =< e, x >$.
- $w(n+1)$ is an immed. subcomp. instruc. of $w(n)$.
- **Lemma** (Moschovakis) The predicate $\{e\}(x)\uparrow$ is Σ_1.

Gandy Selection

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There exists a partial E-REC $g(e, x)$ such that $\forall e\ \forall x$:

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if $g(e,x)\downarrow$, then $g(e,x) \in \omega$ and $\{e\}(\,g(e,x), x)\downarrow$.

E-REC Skolem Functions

Read $w \leq_E z$ as **w** is **E-REC in z**.

And define it by $\exists e[w = \{e\}(z)]$.

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Lemma Suppose $P(x, y)$ is $E - RE$ and

$$\forall x \in z \; \exists y \leq_E x \; \wedge \; P(x, y).$$

Then there exists a partial $E - REC$ f such that

$$\forall x \in z \; f(x) \downarrow \; \wedge \; P(x, f(x)).$$

E-Closed Sets

Say b is **E-closed** iff b is transitive
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Define $\boldsymbol{\kappa^x}$ by $\forall\gamma \quad \gamma < \kappa^x$ iff

$\gamma = \{e\}(x, \overrightarrow{a})$ for some e and $\overrightarrow{a} \in tc(x)$.

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$E(x) = L(\kappa^x, tc(\{x\}))$.

$E(\omega_1)$ is not Σ_1 admissible.

Reflection

$\kappa_0^x = \sup\{\gamma \mid \gamma \leq_E x\}$.

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Say δ is $\mathbf{x}-$**reflecting** iff

$$L(\delta, tc(\{x\})) \models \mathcal{F} \quad \text{implies} \quad L(\kappa_0^x, tc(\{x\})) \models \mathcal{F}$$

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κ_r^x is the greatest x – *reflecting* ordinal.

$\kappa_r^{x,a} \leq \kappa^x$ for all $a \in tc(x)$.

Reflection and Divergence

Assume $x \subset Ord$. Then:

If $\{e\}(x) \uparrow$, then some witness to the divergence

is first order definable over $L(\kappa_r^x, tc(\{x\}))$.

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$E(x)$ is not Σ_1 admissible IFF
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The Divergence-Admissibility Split

Assume $L(\kappa)$ is E-closed. Then (a)$\longleftrightarrow$(b).

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(a) $L(\kappa)$ does not admit divergence witnesses.

(b) $L(\kappa)$ is Σ_1 admissible. And for all $A \subseteq L(\kappa)$:

$$A \text{ is } \Sigma_1^{L(\kappa)} \text{ iff } A \text{ is } E\text{-RE on } L(\kappa).$$

Relativization

Relativizing E-recursion to B means adding a scheme

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$L(\kappa)$ is **E -closed relative to B** iff

$$\{e\}^B(\overrightarrow{x}) \downarrow \longrightarrow \{e\}^B(\overrightarrow{x}) \in L(\kappa)$$

$$\forall e < \omega \quad \forall \overrightarrow{x} \in L(\kappa).$$

Reducibility

Assume $A, B \subseteq L(\kappa)$.

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A is E-reducible to B iff

$$\exists e, p \in L(\kappa) \quad \forall x \in L(\kappa) \quad \{e\}^B(x, p) \downarrow$$

$L(\kappa)$ is E -closed relative to B.

$$x \in A \longleftrightarrow \{e\}^B(x, p) = 1 \qquad (\forall x \in L(\kappa)).$$

Post's Problem

B is **E-RE on L(κ)** IFF $\exists e, p \in L(\kappa)$

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B is **E-RE on $\mathbf{L(\kappa)}$** IFF $\exists e, p \in L(\kappa)$

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Assume $L(\kappa)$ is E-closed.

Then there exist two subsets of $L(\kappa)$,

both E-RE on $L(\kappa)$,

such that neither is E-reducible to the other.

Forcing

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If $\mathcal{P}$ is $c.c.c.$, or countably closed,

and G is $\mathcal{P}$-generic, then $L(\kappa, G)$ is E-closed.

Logic on an E-closed Set

Assume $L(\kappa)$ is E-closed.

$\mathcal{L}$ is an E-REC on $L(\kappa)$ set of atomic symbols.

$\mathcal{L}_{\kappa,\omega}$ is the restriction of $\mathcal{L}_{\infty,,\omega}$ to $L(\kappa)$.

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$\mathcal{F}$ is a sentence of $\mathcal{L}_{\kappa,\omega}$.

$\Delta \vdash \mathcal{F}$ means $\mathcal{F}$ is deducible from Δ in $\mathcal{L}_{\infty,\omega}$.

Deductions

$\Delta \vdash_\kappa \mathcal{F}$ means $\Delta \vdash \mathcal{F}$ via a deduction in $L(\kappa)$.

$\Delta \vdash^E_\kappa \mathcal{F}$ means $\Delta \vdash \mathcal{F}$ via a deduction *E-REC* in $\mathcal{F}$.

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Δ is κ-consistent iff $\Delta \nvdash_\kappa (\mathcal{F} \wedge \neg \mathcal{F})$

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If Δ is κ-consistent, then Δ is $\mathcal{L}_{\infty,\omega}$-consistent.