Low_n Boolean Subalgebras

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Definition 1

The *spectrum of a structure* $\mathcal{A}$ is defined to be the set $\mathrm{Spec}(\mathcal{A})$ of Turing degrees of structures isomorphic to $\mathcal{A}$:

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Question: Must the spectrum of a low$_n$ Boolean algebra always contain the degree **0**?

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$n = 4$ **(Knight & Stob, 2000)**

Every low_4 Boolean algebra is isomorphic to a computable one.

Definition 2

The *spectrum of a relation R on a computable structure $\mathcal{M}$* is defined to be the set $\mathrm{DgSp}_{\mathcal{M}}(R)$ of Turing degrees of all images of R under isomorphisms from $\mathcal{M}$ onto other computable structures:

$$\mathrm{DgSp}_{\mathcal{M}}(R) = \{ \deg(S) \ : \ (\exists \mathcal{B} \leq_T \emptyset) \ [\ (\mathcal{B}, S) \cong (\mathcal{M}, R) \ ] \ \}$$

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We call the computable atomless Boolean algebra $\mathcal{B}$.

$n = 5$ (subalgebra) **(Miller, 2011)**

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"k-fold supremum"

Such an x is called a *k-fold $\mathcal{A}$-supremum* if x is the union of k disjoint single $\mathcal{A}$-suprema.

The property of being a k-fold $\mathcal{A}$-supremum is $\Sigma_4^{\mathcal{A}}$ (uniformly in k).

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Let **c** be a nonlow$_4$ degree and let $C \in \mathbf{c}$. A C-oracle is used to construct a subalgebra $\mathcal{A}$ of $\mathcal{B}$ so that $\mathbf{c} \in \mathrm{DgSp}_{\mathcal{B}}(\mathcal{A})$. $\mathcal{A}$ is built to satisfy:

$$n \in C^{(4)} \iff \exists x \in \mathcal{A}\ [\ x \text{ is a } 2^n\text{-fold } \mathcal{A}\text{-supremum}\ ].$$

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Thus we have $C^{(4)} \leq_T \mathcal{A}^{(4)}$.

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Now, suppose there is a computable subalgebra $\mathcal{D}$ of $\mathcal{B}$ such that $(\mathcal{B}, \mathcal{D}) \cong (\mathcal{B}, \mathcal{A})$.

This implies that $C^{(4)} \leq_T \emptyset^{(4)}$. This would be a contradiction because we chose C to be nonlow$_4$.

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Let f be a total computable function with the property that for all n,

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Then we have $n \in C^{(4)}$ iff $f(n) \in \mathcal{A}^{(4)}$.

Thus $C^{(4)} \leq_1 \mathcal{A}^{(4)}$ via this f.

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Then not only do we have $C^{(4)} \leq_T \mathcal{A}^{(4)}$, we have $C^{(4)} \leq_1 \mathcal{A}^{(4)}$.

fact from the "Jump Theorem"

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upward-closure **(Csima, Harizanov, Miller, Montalbán, 2011)**

If $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ considered as a unary relation on $\mathcal{B}$ which is not intrinsically computable, then $\mathrm{DgSp}_{\mathcal{B}}(\mathcal{A})$ is upwards closed.

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the spectrum of the Boolean algebra

The spectrum of this Boolean algebra as a relation on the computable atomless Boolean algebra is the set of all degrees $\mathbf{d}$ such that

$$C^{(3)} \leq_T D^{(3)} \text{ for some } D \in \mathbf{d}.$$

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We now know that this isn't true for Boolean subalgebras as relations.