Algorithmic randomness and stochastic selection function

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Contents

Algorithmic analogies for classical theorems on subsequences of normal numbers.

- Steinhaus (1922) theorem

$$x \in \mathcal{N} \leftrightarrow \forall 0 < q < 1, u_q\{y \mid x/y \in \mathcal{N}\} = 1.$$

- Kamae (1973) and Weiss (1971) theorem

  Suppose that $\liminf \frac{1}{n} \sum_{i=1}^{n} y_i > 0$ then (i) and (ii) are equivalent:
  (i) $h(y) = 0$.
  (ii) $\forall x \; x \in \mathcal{N} \rightarrow x/y \in \mathcal{N}$,

In van Lambalgen (1987), he showed algorithmic randomness versions and a conjecture to the above theorems. In this talk, I give different algorithmic analogies of above theorems.

Selection function

Example

$$
\begin{aligned}
x &= \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad \cdots \\
y &= \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad \cdots \\
x/y &= \quad 0 \quad \quad 0 \quad \quad 1 \quad \cdots
\end{aligned}
$$

$x, y \in \{0,1\}^\infty$.

$\mathcal{R}^P$: Martin-Löf random sequences w.r.t. P

$(\{0,1\}^\infty, \mathcal{B}, P)$, P: computable.

u_q: (q, 1-q) i.i.d. process, $0 \leq q \leq 1$.

$\mathcal{N}$: normal numbers w.r.t. uniform probability.

Algorithmic analogies for Steinhaus theorem

Theorem 1 (Steinhaus 1922 [3]).

$$x \in \mathcal{N} \leftrightarrow \forall 0 < q < 1, u_q\{y \mid x/y \in \mathcal{N}\} = 1.$$

Theorem 2 (van Lambalgen 1987[6]).
u*: uniform measure on* $\{0,1\}^\infty$*.*
v*: computable probability,* $\forall s \in \{0,1\}^* \;\; v(s0^\infty) = 0$*.*

$$\{(x,y) \mid x/y \in \mathcal{R}^u\} \supseteq \mathcal{R}^{u \times v}.$$

$$v\{y \mid x/y \in \mathcal{R}^u\} = 1 \text{ for } x \in \mathcal{R}^u.$$

From theorem 2,

$$\text{(i) } x \in \mathcal{R}^u \to \text{(ii)} \forall y \; (x,y) \in \mathcal{R}^{u \times v} \to \; x/y \in \mathcal{R}^u$$

(i) $\leftarrow$ (ii) is trivial if we consider $y = 11\cdots$ and $v(y) = 1$.
y is called non-trivial if y contains infinitely many 1s and 0s.

ML-randomness analogies for Steinhaus theorem

Proposition 1. *There is a non-trivial computable selection function y such that the following two statements are equivalent.*
(i) $x \in \mathcal{R}^P$, *P is comp. ergodic prob..*
(ii) $x \in \cup_{\text{comp ergodic } Q} \mathcal{R}^Q \; x/y \in \mathcal{R}^{P|y}$

$P|y$: the marginal distribution of random variables selected by y.

Corollary 1. *There is a non-trivial selection function y such that the following two statements are equivalent.*
(i) $x \in \mathcal{R}^u$, *u is uniform prob..*
(ii) $x \in \cup_{\text{comp ergodic } Q} \mathcal{R}^Q \; x/y \in \mathcal{R}^u$

Problem 1. *Show the equivalence of the following two statements.*

(i) $x \in \mathcal{R}^u$.

(ii) $\exists$ *computable* w $\;x \in \mathcal{R}^w$ *and* $x/y \in \mathcal{R}^u$ *for* $y \in Y_{w,x}$
where $\{Y_{w,x}\}$ *consists of non-trivial selection functions*
and depends on w *and* x.

Complexity rate

K : prefix complexity

Definition 1. y has maximal complexity rate with respect to a computable P if

$$\lim_{n\to\infty} K(y_1^n)/n = \lim_{n\to\infty} -\frac{1}{n}\log P(y_1^n). \tag{1}$$

For example, y has maximal complexity rate with respect to (1/2,1/2)-i.i.d. process if

$$\lim_{n\to\infty} K(y_1^n)/n = 1.$$

y is Martin-Löf random w.r.t. some computable ergodic $P \Rightarrow$ (1).
(due to upcrossing inequality for SMB theorem, Hochman 2009 [1]).

Complexity ratio version of Steinhaus theorem

Proposition 2. *Let w be a computable probability such that*

(a) $\forall y \in \mathcal{R}^w,\ \lim_n K(y^n)/n = 0$,
(b) $\lim_n \sum_{1\le i\le n} y_i/n > 0$ *exists for* $y \in \mathcal{R}^w$, *and*
(c) $\forall 1 \ge \epsilon > 0 \exists y \in \mathcal{R}^w \lim_n \sum_{1\le i\le n} y_i/n > 1-\epsilon$.

Then the following two statements are equivalent.

(i) $\lim_{n\to\infty} \frac{1}{n} K(x^n) = 1$.
(ii) $\lim_{n\to\infty} \frac{1}{|x^n/y^n|} K(x^n/y^n) = 1$ *for* $y \in \mathcal{R}^w$, *where* K *is the prefix-complexity.*

Example: Let P_ρ be a probability derived from irrational rotation with parameter ρ (Sturmian sequences) and $w := \int P_\rho d\rho$.
Then w satisfies the condition of Prop. 2, see Takahashi and Aihara [5].

Kamae-Weiss theorem

Definition 2. p is called cluster point if there is a sequence $\{n_i\}$

$$\forall s \; p(s) = \lim_{i\to\infty} \#\{1 \leq j \leq n_i \mid x_j \cdots x_{j+|s|-1} = s\}/n_i.$$

Let $V(x)$ be the set of cluster point of x. $V(x) \neq \emptyset$ for all x.
Kamae entropy is defined by

$$h(x) = \sup\{h(p) \mid p \in V(x)\}.$$

Theorem 3 (Kamae[2])**.** *Suppose that* $\liminf \frac{1}{n} \sum_{i=1}^{n} y_i > 0$ *then (i) and (ii) are equivalent:*
(i) $h(y) = 0$*.*
(ii) $\forall x, \; x \in \mathcal{N} \to x/y \in \mathcal{N}$*,*

Note: The part (i)$\Rightarrow$ (ii) is appeared in Weiss [7].

van Lambalgen's conjecture

In van Lambalgen [6] the following equivalence is conjectured:

(i) $\lim_{n\to\infty} K(y_1^n)/n = 0$.

(ii) $\forall x,\ x \in \mathcal{R} \to x/y \in \mathcal{R}$.

Proposition 3. *Suppose that* y *is Martin-Löf random with respect to some computable probability* P *and* $\sum_{i=1}^{\infty} y_i = \infty$*. Then the following two statements are equivalent:*

(i) y *is computable.*

(ii) $\forall x,\ x \in \mathcal{R} \to\ x/y \in \mathcal{R}^y$*.*

$\mathcal{R}^y$: the set of Martin-Löf random sequences with respect to (1/2,1/2)-i.i.d. process relative to y.

For a proof, see Takahashi [4].

Proposition 4. *Suppose that*
y *has maximal complexity rate with respect to some computable measure and*
$\lim_n \frac{1}{n}\sum_{i=1}^n y_i > 0$.
Then the following two statements are equivalent:

(i) $\lim_{n\to\infty} K(y_1^n)/n = 0$.

(ii)
$\forall x \ \lim_{x\to\infty} K(x_1^n)/n = 1 \rightarrow \ \lim_{n\to\infty} \frac{1}{|x_1^n/y_1^n|} K(x_1^n/y_1^n|y_1^n) = 1.$

Example of y : computable sequence, sturmian sequence.

For a proof, see Takahashi [4].

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