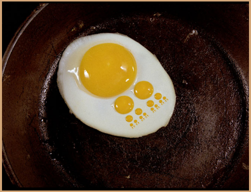
Translating the Cantor set by a random



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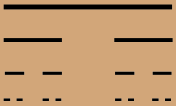
Overview

Definition

The *Cantor set* $\mathcal{C} \subseteq [0,1]$ consists of all real numbers whose tertiary expansion contains only 0's and 2's.



Question

Can points in the Cantor set cancel randomness when added to a Martin-Löf random real r?

- Yes! Many points in $\mathcal{C}$ cancel randomness.
- Effective level sets of $\mathcal{C} + r$ are classical fractals.
- The reason seems to involve additive number theory.

Classical Hausdorff dimension

For $E \subseteq \mathbb{R}$,

- $|E|$ is the *diameter* of E,
- A class $\mathcal{G}$ *covers* E if its union contains E and,
- $\mathcal{G}$ a δ*-mesh* if every member of $\mathcal{G}$ has diameter at most δ.

For $\beta \geq 0$, let

$$\mathcal{H}_\delta^\beta(E) = \inf \left\{ \sum_{G \in \mathcal{G}} |G|^\beta : \mathcal{G} \text{ is a countable } \delta\text{-mesh cover of } E \right\}.$$

The β*-dimensional Hausdorff measure* of E is $\lim_{\delta \to 0} \mathcal{H}_\delta^\beta(E)$.

Definition

The *Hausdorff dimension of* $E \subseteq \mathbb{R}$, or $\dim_{\mathrm{H}}(E)$, equals the unique β where $\mathcal{H}^\beta(E)$ transitions from ∞ to 0.

The correct notion of dimension

Definition

The *effective Hausdorff dimension* of a set $E \subseteq R$, or $\mathrm{cdim}_{\mathrm{H}}(E)$, is the same as its classical counterpart except covers are restricted to effectively open classes.

Equivalently, we can define in terms of Kolmogorov complexity K.

Theorem (Levin, Lutz, Mayordomo)

The constructive dimension of a singleton $x \in \mathbb{R}$ is

$$\mathrm{cdim}_{\mathrm{H}}\{x\} = \liminf_{n \to \infty} \frac{K(x \upharpoonright n)}{n}.$$

Furthermore,

$$\mathrm{cdim}_{\mathrm{H}}(E) = \sup\{\mathrm{cdim}_{\mathrm{H}}\{x\} : x \in E\}.$$

Dimension facts

The following are true.

- For any Martin-Löf random real x,
  - $\dim_{\mathrm{H}}\{x\} = 0$ but
  - $\mathrm{cdim}_{\mathrm{H}}\{x\} = 1$.

  The converse does not hold.
- A real consisting of a Martin-Löf random real interleaved with a computable real has constructive dimension $1/2$.

Theorem (Lutz, Cai and Hartmanis)

The set of reals with constructive dimension α has both effective and classical Hausdorff dimension α.

Properties of the Cantor set

Definition

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

1. The dimension of the Cantor set is:

$$\dim_{\mathrm{H}}(\mathcal{C}) = \mathrm{cdim}_{\mathrm{H}}(\mathcal{C}) = \frac{\log 2}{\log 3} \approx 0.6309.$$

   Packing dimension, upper box counting, and lower box counting also agree with this value.

2. Any real in the interval $[0, 2]$ can be written as the sum of two members of the Cantor set. Indeed, $\frac{1}{2}\mathcal{C} + \frac{1}{2}\mathcal{C} = [0, 1]$. Thus adding two numbers from the Cantor set suffices to completely cancel randomness.

Translating the Cantor set by a random

Definition

Let $\mathcal{E}_{=\alpha}$ be the set of reals whose constructive dimension is α, and let

$$\mathcal{C} + r = \{x + r : x \in \mathcal{C}\}.$$

Our main result is the following:

Theorem

For any α *satisfying* $1 - \dim_{\mathrm{H}} \mathcal{C} \leq \alpha \leq 1$, *and for any Martin-Löf random* $r \in [0, 1]$, *we have*

$$\dim_{\mathrm{H}} [(\mathcal{C} + r) \cap \mathcal{E}_{=\alpha}] = \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}.$$

Thus the difference between the effective and classical Hausdorff dimensions of this intersection is $1 - \dim_{\mathrm{H}} \mathcal{C}$.

Why would this be true?

Definition

Let $\mathcal{E}_{\leq\alpha}$ denote the set of reals with effective Hausdorff dimension at most α.

Let

$$B_x = (\mathcal{C} + x) \cap \mathcal{E}_{\leq\alpha}$$

and let $\gamma > \dim_{\mathrm{H}} \mathcal{C} + \alpha$. B_x is the vertical fiber of a set B with $\mathrm{cdim}_{\mathrm{H}}(B) < \gamma$, so by the Fubini inequality for Hausdorff measures

$$\int \mathcal{H}^{\gamma-1}(B_x)\, d\mathcal{H}^1(x) \leq c\mathcal{H}^{\gamma}(B) = 0$$

for some $c > 0$, whence

$$\dim_{\mathrm{H}}(B_x) \leq \gamma - 1 = \alpha - (1 - \dim_{\mathrm{H}} \mathcal{C})$$

for Lebesgue measure almost all x.

A point within $2/3$ of optimal

A quick calculation gives the best possible randomness cancellation.

Proposition

Let $x \in \mathcal{C}$ and let r be a Martin-Löf random real. Then

$$1 - \dim_{\mathrm{H}} \mathcal{C} \leq \mathrm{cdim}_{\mathrm{H}}\{x + r\} \leq 1.$$

Example

Let $r \in [0, 1]$ be a real with ternary expansion $.r_1 r_2 \ldots$. Choose $t = .t_1 t_2 \ldots \in \mathcal{C}$ as follows. Let

$$t_n = \begin{cases} 0 & \text{if } r_n \in \{1, 2\}, \\ 2 & \text{otherwise.} \end{cases}$$

A point within 2/3 of optimal II

Example (cont.)

Then

$$r_n + t_n = \begin{cases} 2 & \text{if } r_n = 0, \\ 1 & \text{if } r_n = 1, \\ 2 & \text{if } r_n = 2, \end{cases}$$

and so by a theorem of Lutz,

$$\begin{aligned} \mathrm{cdim_H}\{r+t\} &\leq \text{entropy}\left(0, \frac{1}{3}, \frac{2}{3}\right) \\ &= -\frac{1}{3}\log_3\frac{1}{3} - \frac{2}{3}\log_3\frac{2}{3} = 1 - \frac{2}{3}\cdot \dim_{\mathrm{H}} \mathcal{C} \approx 0.5793, \end{aligned}$$

which is within 2/3 of optimal cancellation.

Can we do better?

The method from the previous slide resists improvement. Suppose there were a set of reals E such that:

- E has low effective Hausdorff dimension, and
- $\mathcal{C} + E \supseteq [0, 2]$.

Then for any $r \in [0, 2]$, there exist $x \in \mathcal{C}$ and $y \in E$ such that $x + y = 2 - r$, so $x + r \in 2 - E$, and therefore

$$\text{cdim}_{\mathrm{H}}\{x + r\} \leq \text{cdim}_{\mathrm{H}} E.$$

Question

How small can $\text{cdim}_{\mathrm{H}} E$ be?

Building blocks

Remark

$\mathcal{C} + E \supseteq [0,2] \iff \frac{1}{2}\mathcal{C} + \frac{1}{2}E \supseteq [0,1].$

Example

Let $C_2 = \{00, 01, 10, 11\}$. Then the ternary expansion is:

$$\frac{1}{2}\mathcal{C} = \{.x_0x_1x_2\cdots : x_i \in C_2\}.$$

Let $B_2 = \{00, 02, 11\}$, and let $\frac{1}{2}E = \{.x_0x_1x_2\cdots : x_i \in B_2\}$. Then the condition in "Remark" holds:

$$\begin{aligned} 00 &= 00 + 00, & 10 &= 10 + 00, & 20 &= 11 + 02, \\ 01 &= 01 + 00, & 11 &= 11 + 00, & 21 &= 10 + 11, \\ 02 &= 00 + 02, & 12 &= 01 + 11, & 22 &= 11 + 11. \end{aligned}$$

Effective Hausdorff dimension with B_2

In the example on the previous slide, $B_2 = \{00, 02, 11\}$ and

$$E' = \frac{1}{2}E = \{.x_0 x_1 x_2 \cdots : x_i \in B_2\}$$

is the additive "complement" to the Cantor set. For any $\beta \geq 0$, we can uniformly cover E' with $|B_2|^n$ intervals of size $3 \cdot 3^{-2n}$:

- there are $|B_2|$ block choices for each of the first n blocks in any member of E', and
- a closed interval of length $3 \cdot 3^{-2n}$ covers all possible extensions of the n-block prefix.

The $1/2$-effective Hausdorff measure of E' is

$$c\mathcal{H}^{1/2}(E') \leq \lim_{n \to \infty} |B_2|^n \cdot [3 \cdot (3^{-2n})]^{1/2} = \sqrt{3},$$

so $\mathrm{cdim}_{\mathrm{H}}(E') \leq 0.5$.

Optimal blocks

Might we decrease the effective Hausdorff dimension of E by using larger blocks? The following blocks are optimal, but not unique.

$$
\begin{aligned}
B_1 &= \{0, 1\}, \\
B_2 &= \{00, 02, 11\}, \\
B_3 &= \{000, 002, 021, 110, 112\}, \\
B_4 &= \{0000, 0002, 0011, 0200, 0202, 0211, 1100, 1102, 1111\}, \\
B_5 &= \{00000, 00002, 00021, 00112, 00210, 01221, 02012, \\
&\quad 02110, 02201, 10212, 11010, 11101, 11120, 11122\}
\end{aligned}
$$

Note that $B_4 = B_2 \times B_2$. Unknown: is $B_2 \times B_4$ optimal for B_6?

Question

Can you beat B_3 ($\log 5 / \log 27 \approx 0.4883$)?

I am your density

Definition

For any set of positive integers A,

$$\text{density}(A) = \lim_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.$$

Blocks by Lorentz

Lorentz's Lemma (1954)

There exists a constant c such that for any integer k, if $A \subseteq [0, k)$ is a set of integers with $|A| \geq \ell \geq 2$, then there exists a set of integers $B \subseteq (-k, k)$ such that $A + B \supseteq [0, k)$ with $|B| \leq ck\frac{\log \ell}{\ell}$.

Lorentz's proof solves a conjecture of Paul Erdős: every infinite set A has a complementary set B of density zero.

Theorem

There exists a uniform sequence of computably closed sets $E_1, E_2, \ldots$ such that

1. $\frac{1}{2}\mathcal{C} + \frac{1}{2}E_n = [0, 1]$ *for all n, and*
2. $\lim_{n\to\infty} \operatorname{cdim}_{\mathrm{H}} E_n = 1 - \dim_{\mathrm{H}} \mathcal{C}$.

Blocks by Kolmogorov

Theorem (Lower Bound)

Let $1 - \dim_{\mathrm{H}} \mathcal{C} \leq \alpha \leq 1$ *and let* r *be any real in* $[0, 1]$. *Then*

$$\dim_{\mathrm{H}} [(\mathcal{C} + r) \cap \mathcal{E}_{\leq \alpha}] \geq \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}.$$

Let $\mathcal{C}_A$ be the set of reals having a ternary expansion whose digits are all 0 or 2, where the 2's only occur at positions in

$$A = \{\lfloor y/D \rfloor : y \in \mathbb{N}\},$$

which has density $D = \frac{1-\alpha}{\dim_{\mathrm{H}} \mathcal{C}}$ and low Kolmogorov complexity.

Lemma (Density Style)

Let α *be as above. Then there exists a computably closed set of reals* E *such that* $\operatorname{cdim}_{\mathrm{H}} E = \alpha$ *and* $\mathcal{C}_A + E = [0, 2]$.

Proof of Lower Bound

Let $\mathcal{C}_{\bar{A}}$ denote the set of reals whose ternary expansions consists of only 0's and the 2's only occur at positions not in A. For each $z \in \mathcal{C}$ there exist unique reals $v \in \mathcal{C}_A$ and $w \in \mathcal{C}_{\bar{A}}$ such that $v + w = z$; let $p(z) = w$. Take E with $\operatorname{cdim}_{\mathrm{H}} E = \alpha$ as in the Density Style Lemma and let $F = 2 - E$ so that $F - \mathcal{C}_A = [0, 2]$. Let $S = \mathcal{C} \cap (F - r)$.

Fix $r \in [0, 1]$. For each $y \in \mathcal{C}_{\bar{A}}$ we have $r + y \in F - \mathcal{C}_A$, so there exists $x \in \mathcal{C}_A$ such that $r + y \in F - x$, which gives $x + y \in S$ since $\mathcal{C}_A + \mathcal{C}_{\bar{A}} = \mathcal{C}$. Thus p maps S onto $\mathcal{C}_{\bar{A}}$. Since p is Lipschitz we get, by proper choice of density D,

$$\dim_{\mathrm{H}} S \geq \dim_{\mathrm{H}} \mathcal{C}_{\bar{A}} \geq \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}$$

because Lipschitz maps do not increase Hausdorff dimension. QED

Open problems

Theorem

For any α *satisfying* $1 - \dim_{\mathrm{H}} \mathcal{C} \leq \alpha \leq 1$, *and for any Martin-Löf random* $r \in [0,1]$, *we have*

$$\dim_{\mathrm{H}} [(\mathcal{C} + r) \cap \mathcal{E}_{=\alpha}] = \dim_{\mathrm{H}} [(\mathcal{C} + r) \cap \mathcal{E}_{\leq\alpha}] = \alpha - 1 + \dim_{\mathrm{H}} \mathcal{C}.$$

Questions

- How much can the randomness of r be reduced by adding a Cantor set point if r was not completely random to begin with?
- Given a real x, what is the least possible dimension for $\{x + y\}$ when y is Martin-Löf random? When $y \in \mathcal{C}$?
- What about fractals other than the Cantor set?
- What about other measures of dimension?

Simple bounds

back to 2/3 construction

Theorem

Let $x \in \mathcal{C}$ and let r be a Martin-Löf random real. Then $1 - \dim_{\mathrm{H}} \mathcal{C} \leq \mathrm{cdim}_{\mathrm{H}}\{x + r\} \leq 1$.

Proof.

Given x and r, there is a constant c such that for all n, $K[(x+r) \upharpoonright n] + K(x \upharpoonright n) + c \geq K(r \upharpoonright n)$. Thus,

$$\begin{aligned}
1 \geq \mathrm{cdim}_{\mathrm{H}}\{x + r\} &= \liminf_{n\to\infty} \frac{K[(x+r)\upharpoonright n]}{n} \\
&\geq \liminf_{n\to\infty} \frac{K(r \upharpoonright n)}{n} - \limsup_{n\to\infty} \frac{K(x \upharpoonright n)}{n} \\
&\geq 1 - \mathrm{cdim}_{\mathrm{P}}\{x\} \geq 1 - \mathrm{cdim}_{\mathrm{P}}\,\mathcal{C} = 1 - \dim_{\mathrm{H}} \mathcal{C}.
\end{aligned}$$

□