On the Strongly Bounded Turing Degrees of C.E. Sets: Degrees Inside Degrees

Klaus Ambos-Spies

University of Heidelberg
Institut für Informatik

The Incomputable

Chicheley Hall - 15 June 2012

Strong Reducibilities 1: Truth-Table Type Reducibilities

- Though Turing reducibility plays a distinguished role by formalizing relative computability in the general sense, most of the reductions encountered in applications are of special simpler forms. So already Post (1944) introduced strong reducibilities capturing some of the common reductions.

- In a strong reducibility the access to or the use of the oracle is limited. In the reducibilities introduced by Post the number of queries may be bounded, the queries are nonadaptive (oracle independent), and the evaluation strategy is specified ahead (i.e., oracle independent too):
  - many-one (m): 1 query - positive
  - bounded truth-table (btt): a constant number of queries - any truth-table
  - truth-table (tt): unbounded number of queries - any truth table

Strong Reducibilities 2: (Strongly) Bounded Turing Reducibilities

In contrast to the truth-table-type reducibilities of Post, the bounded Turing reducibilities are obtained by imposing bounds $b(x)$ on the oracle queries in a Turing reduction $A(x) = \Phi^B(x)$ but not limiting the evaluation process:

- $b(x)$ computable: weak truth-table (wtt) (or bounded Turing (bT))
- $b(x) = id(x) = x$: identity bounded Turing (ibT)
- $b(x) = id(x) + c = x + c$: computable Lipschitz (cl) (or strong weak truth-table (sw))

In the following we refer to ibT and cl as the strongly bounded Turing (sbT) reducibilities.

Origins and Properties

- cl-Reducibility was introduced by Downey, Hirschfeldt and LaForte (2001) in the context of computable randomness. The special case of ibT-reducibility was introduced by Soare (2004) in the context of some applications of computability theory to differential geometry (Nabutovski and Weinberger).

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- cl-Reducibility preserves Kolmogorov complexity: For a set A which is cl-reducible to a set B, the Kolmogorov complexity of $A \upharpoonright n$ is bounded by the Kolmogorov complexity of $B \upharpoonright n$ up to an additive constant.

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- For $r = \mathrm{ibT}, \mathrm{cl}$, r-reducibility is a preordering, closed under finite variants, but not computably invariant.

Examples of ibT-reductions on the c.e. sets

Some typical, frequently used examples of ibT-reductions on the c.e. sets are the following.

- Splitting

$$A = B \dot{\cup} C \;\Rightarrow\; B \leq_{\mathrm{ibT}} A \text{ and } C \leq_{\mathrm{ibT}} A$$

  In fact, $deg_r(A) = deg_r(B) \vee deg_r(C)$ for $r = \mathrm{ibT}, \mathrm{cl}$.

- Permitting

$$x \in A_{at\ s} \;\Rightarrow\; \exists\, y \leq x\, (y \in B_{at\ s})$$

  In fact, any ibT-reduction $A \leq_{\mathrm{ibT}} B$ among c.e. sets may be represented by a permitting reduction (by replacing A and B by appropriate ibT-equivalent subsets).

Note that neither of the above reductions is a truth-table reduction.

Strongly Bounded Turing Degrees of C.E. Sets

- The study of the partial orderings $(\mathbf{R}_{\mathrm{ibT}}, \leq)$ and $(\mathbf{R}_{\mathrm{cl}}, \leq)$ of the strongly bounded Turing degrees of the computably enumerable sets is a fairly new subject.

- The investigations use the methods known from the Turing degrees (in particular priority arguments) but also some techniques specific for the sbT-degrees (e.g. shift and density arguments).

  Moreover some transfer techniques have been developed which allow to transfer certain results on the wtt-degrees to the sbT-degrees (though these degree structures look quite different).

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- $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ does not have maximal (hence no greatest) elements (Barmpalias 05). Moreover, there are *maximal pairs*, i.e., c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ s.t. there is no c.e. degree $\mathbf{c} \geq \mathbf{a}, \mathbf{b}$ (Barmpalias 05; Fan and Lu 05); so, in particular, $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ is not an upper semi-lattice.

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- All countable distributive lattices can be embedded into $(\mathrm{R_{sbT}}, \leq)$ (preserving 0) and so can the two 5-element nondistributive lattices N_5 and M_3 (though M_3 cannot be embedded preserving 0) (A-S, Bodewig, Kräling, Yu; A-S, Bodewig, Kräling, Wang).

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- Every degree $\mathbf{a} \in \mathrm{R}_{\mathrm{sbT}} \setminus \{\mathbf{0}\}$ has the anti-cupping property (A-S, Wang).

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The analysis of the power of such transfer techniques has led us to study the c.e. cl-degrees inside a single c.e. wtt-degree and the c.e. ibT-degrees inside a single c.e. cl-degree.

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  - greatest element = least element (= contiguous)
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  - a least element but not a greatest element
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- In general, $(\mathbf{R}_{\mathrm{wtt}}(deg_{\mathrm{T}}(A)), \leq)$ is not a lattice and not closed under meets (Downey) - though there are some $(\mathbf{R}_{\mathrm{wtt}}(deg_{\mathrm{T}}(A)), \leq)$ which are infinite distributive lattices (P.Fischer ?).

C.E. cl-(ibT-)Degrees Inside a Single C.E. wtt-(cl-)Degree

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- the p.o. $(\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ of the c.e. ibT-degrees inside the cl-degree of A
- the p.o. $(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ of the c.e. cl-degrees inside the wtt-degree of A

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Basic questions:

- Does (the theory of) $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ and $(\mathrm{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(A)), \leq)$ depend on the choice of A?
- How simple can $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ and $(\mathrm{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(A)), \leq)$ be (a singleton, finite, a linear ordering, a distributive lattice)?

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Bounded-Shift Lemma. Let A be any noncomputable c.e. set and, for $k \geq 1$ let $A + k = \{x + k : x \in A\}$. Then $A + k <_{\mathrm{ibT}} A$ and $A + k =_{\mathrm{cl}} A$.

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Bounded-Shift Inversion Lemma. Let A be any noncomputable c.e. set and let $k \geq 1$. There is a c.e. set B such that $A =_{\mathrm{ibT}} B + k$ (namely $B = A - k = \{x - k : x \geq k \,\&\, x \in A\}$).

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So $A + 1 <_{\mathrm{ibT}} A <_{\mathrm{ibT}} A - 1$ while $A =_{\mathrm{cl}} A + 1 =_{\mathrm{cl}} A - 1$. Hence $(\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ has neither minimal nor maximal elements.

$(\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$: chains

THEOREM 1. For any noncomputable c.e. set A, the partial ordering $(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ possesses an infinite chain of degrees

$$(*) \quad \cdots < \mathbf{a}_{-1} < \mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \ldots$$

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PROOF (IDEA). Let $\mathbf{a}_{-k}$ and $\mathbf{a}_k$ be the ibT-degrees of $A + k$ and $A - k$ respectively. Then $(*)$ follows from the Bounded-Shift Lemma and the Bounded-Shift Inversion Lemma. The remaining parts of the theorem hold by the following observations: (1) for any $B =_{\mathrm{cl}} A$ there is a k such that $A + k \leq_{\mathrm{ibT}} B \leq_{\mathrm{ibT}} A - k$ and (2) for any $k \geq 0$ the bounded shift $A \to A + k$ induces an automorphism of $(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$.

$(\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$: chains (2)

COROLLARY. No finite chain in $(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ is maximal.

$(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$: chains (2)

COROLLARY. No finite chain in $(\mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ is maximal.

COROLLARY. The partial ordering $(\mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ is upward-directed, i.e.,

$$\forall\, \mathbf{a}, \mathbf{b} \in \mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A))\ \exists\, \mathbf{c} \in \mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A))\ (\mathbf{a}, \mathbf{b} \leq \mathbf{c}),$$

and downward-directed, i.e.,

$$\forall\, \mathbf{a}, \mathbf{b} \in \mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A))\ \exists\, \mathbf{c} \in \mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A))\ (\mathbf{c} \leq \mathbf{a}, \mathbf{b}),$$

So, in particular, $(\mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ is closed under joins and meets (whenever defined).

$(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$: minimal and maximal elements

The argument that $(\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ does not contain minimal elements carries over to $(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ by the observation of Downey, Hirschfeldt and LaForte that, for noncomputable A,

$$2A <_{\mathrm{cl}} A$$

while, obvioulsy, $2A =_{\mathrm{wtt}} A$ (where $2A = \{2x : x \in A\}$).

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For a proof of the dual result we have to look at computable shifts more generally.

A computable shift f is a stricty increasing computable function. A shift f is unbounded if $f(x) - x \to \infty$.

Notation: $A_f = f(A) = \{f(x) : x \in A\}$ "f-shift of A"

$(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$: minimal and maximal elements (2)

Computable-Shift Lemma (A-S, Ding, Fan, Merkle). Let A be a noncomputable c.e. set and let f be a computable unbounded shift. Then $A_f <_{\mathrm{ibT}} A$ and $A_f <_{\mathrm{cl}} A$ while $A_f =_{\mathrm{wtt}} A$.

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Computable-Shift Inversion Lemma (A-S, Ding, Fan, Merkle; Belanger). Let A be any noncomputable c.e. set. There is a c.e. set B and *some* unbounded computable shift f such that $A =_{\mathrm{ibT}} B_f$.

$(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$: minimal and maximal elements (2)

Computable-Shift Lemma (A-S, Ding, Fan, Merkle). Let A be a noncomputable c.e. set and let f be a computable unbounded shift. Then $A_f <_{\mathrm{ibT}} A$ and $A_f <_{\mathrm{cl}} A$ while $A_f =_{\mathrm{wtt}} A$. Moreover, for any c.e. set B such that $A_f \cap B = \emptyset$ and $A \leq_{\mathrm{cl\ (ibT)}} A_f \cup B$, $A \leq_{\mathrm{cl\ (ibT)}} B$.

Computable-Shift Inversion Lemma (A-S, Ding, Fan, Merkle; Belanger). Let A be any noncomputable c.e. set. There is a c.e. set B and *some* unbounded computable shift f such that $A =_{\mathrm{ibT}} B_f$.

So $A_f <_{\mathrm{cl}} A$ while $A =_{\mathrm{wtt}} A_f$ for *any* unbounded computable shift; and $A =_{\mathrm{cl}} B_f <_{\mathrm{cl}} B$ while $A =_{\mathrm{wtt}} B$ for *some* unbounded computable shift f (and some B).

Hence $(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ has neither minimal nor maximal elements.

$(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$: chains

THEOREM 2. For any noncomputable c.e. set A, the partial ordering $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ possesses an infinite chain of degrees

$$(*) \quad \cdots < \mathbf{a}_{-1} < \mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \dots$$

of order type $\mathbb{Z} = \omega^* + \omega$ such that $\mathbf{a}_0 = deg_{\mathrm{cl}}(A)$

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In contrast to Theorem 1 here the chain $(*)$ is not effectively given and, in general, we cannot ensure that

$$\forall\, \mathbf{b} \in \mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A))\ \exists\, k > 0\ [\mathbf{b} < \mathbf{a}_k].$$

Namely, A-S, Ding, Fan and Merkle have shown that there is a maximal pair of cl-degrees in the wtt-degree of K.

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So, in particular, $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ is closed under meets (whenever defined).

For some A, however, $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ is not upward-directed.

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For some A, however, $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ is not upward-directed.

Still, by a more sophisticated argument, we can show that, for any A, $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ is closed under joins (whenever defined).

$(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$: anti-chains

THEOREM 3. No finite anti-chain in $(\mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ is maximal.

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PROOF (IDEA). We show that there is a c.e. set B such that A and B are ibT-incomparable and cl-equivalent.

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Let $A_{even} = A \cap 2\mathbb{N}$ and $A_{odd} = A \cap 2\mathbb{N} + 1$.

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- Case 2: Otherwise

  Then w.l.o.g. we may assume that $A = A_{even} \subseteq 2\mathbb{N}$. By Sacks' Splitting split A into T-incomparable sets A_0 and A_1 and let $B = A_0 - 1 \cup A_1 + 2$.

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THEOREM 4. Any finite partial ordering can be embedded into $(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$.

$(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$: anti-chains

Here we obtain the corresponding results (even in a somewhat stronger form):

THEOREM 5. No finite anti-chain in $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ is maximal.

THEOREM 6. Any finite (in fact countable) distributive lattice can be embedded into $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$.

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COROLLARY. For any noncomputable c.e. sets A and B,

$$\exists\text{-}\mathrm{Th}(\mathbf{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq) = \exists\text{-}\mathrm{Th}(\mathbf{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(B)), \leq).$$

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In contrast to Theorem 4 the proof of Theorem 6 is uniform. It uses the existence of sufficiently "scattered" sets in any c.e. wtt-degree. The embedding does not require the minimal pair technique (finite injury).

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In contrast to Theorem 4 the proof of Theorem 6 is uniform. It uses the existence of sufficiently "scattered" sets in any c.e. wtt-degree. The embedding does not require the minimal pair technique (finite injury).

By considering sufficiently well behaved A we can use similar arguments to show that $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ itself is a distributive lattice (and similarly for $(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$).

The structure of $(\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ for scattered A.

DEFINITION. A c.e. set A is scattered if

$$\forall\, k \geq 1\ \forall^{\infty}\, n\ (|A \cap [n, n+k]| \leq 1).$$

(Note that, for any c.e. set A, $A^2 = \{x^2 : x \in A\}$ is scattered and $A^2 =_{\mathrm{wtt}} A$. But there are c.e. sets which are not cl-equivalent to any scattered set.)

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THEOREM 7. Let A be a scattered noncomputable c.e. set. $(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ is a dense distributive lattice.

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THEOREM 7. Let A be a scattered noncomputable c.e. set. $(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ is a dense distributive lattice. Moreover, for any degrees $\mathbf{a}, \mathbf{b} \in \mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A))$ such that $\mathbf{b} < \mathbf{a}$, the countable atomless Boolean algebra can be embedded into the interval $[\mathbf{b}, \mathbf{a}]$ (preserving 0 and 1) but the interval $[\mathbf{b}, \mathbf{a}]$ is not complemented.

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The structure of $(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ for hyper-scattered A.

DEFINITION. A c.e set A is hyper-scattered if, for any computable shift f, there is a computable subset B of A such that

$$\forall\ n \geq 0\ (|(A \setminus B) \cap [n, f(n)]| \leq 1).$$

(There are noncomputable hyper-scattered sets but not any c.e. wtt-degree contains a hyper-scattered set. For instance the wtt-degree of any hyper-scattered set does not contain any array noncomputable set.)

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THEOREM 8. Let A be a hyper-scattered noncomputable c.e. set. $(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ is a dense distributive lattice. Moreover, for any degrees $\mathbf{a}, \mathbf{b} \in \mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A))$ such that $\mathbf{b} < \mathbf{a}$, the countable atomless Boolean algebra can be embedded into the interval $[\mathbf{b}, \mathbf{a}]$ (preserving 0 and 1) but the interval $[\mathbf{b}, \mathbf{a}]$ is not complemented.

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The structure of $(\mathrm{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(A)), \leq)$ for certain A.

Most of the above properties of the partial orderings $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ and $(\mathrm{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(A)), \leq)$ for scattered A and hyper-scattered A, respectively, do not hold for all A:

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- For some A, $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ and $(\mathrm{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(A)), \leq)$ are neither an upper semilattice nor a lower semilattice.
- For some A, $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ and $(\mathrm{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(A)), \leq)$ are not distributive. In fact, for some A, N_5 can be embedded into $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ and, for some A, N_5 and M_3 can be embedded into $(\mathrm{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(A)), \leq)$.
- For some A, $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ and $(\mathrm{R}_{\mathrm{cl}}(\mathit{deg}_{\mathrm{wtt}}(A)), \leq)$ are not dense.

These results either follow directly from the corresponding results for $(\mathrm{R}_{\mathrm{ibT}}, \leq)$ and $(\mathrm{R}_{\mathrm{cl}}, \leq)$ or obtained by refining the techniques introduced there.

Further Results: Interpolation

As pointed out before, any degree in $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ is branching and any nonzero degree in $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ splits. One might try to carry over these results to the p.o. $(\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ and $(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ for all noncomputable c.e. sets A by proving the following interpolation phenomena (for all noncomputable c.e. sets A and B):

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(i) $(A <_{\mathrm{ibT}} B \;\&\; A \neq_{\mathrm{cl}} B) \;\Rightarrow\; \exists\, C =_{\mathrm{cl}} B\, (A <_{\mathrm{ibT}} C <_{\mathrm{ibT}} B)$
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(ii) $(A <_{\mathrm{cl}} B \;\&\; A \neq_{\mathrm{wtt}} B) \;\Rightarrow\; \exists\, C =_{\mathrm{wtt}} B\, (A <_{\mathrm{cl}} C <_{\mathrm{cl}} B)$
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Further Results: Interpolation

As pointed out before, any degree in $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ is branching and any nonzero degree in $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ splits. One might try to carry over these results to the p.o. $(\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ and $(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ for all noncomputable c.e. sets A by proving the following interpolation phenomena (for all noncomputable c.e. sets A and B):

(i) $(A <_{\mathrm{ibT}} B \;\&\; A \neq_{\mathrm{cl}} B) \;\Rightarrow\; \exists\, C =_{\mathrm{cl}} B \,(A <_{\mathrm{ibT}} C <_{\mathrm{ibT}} B)$
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But only (ii) holds in general. So, by interpolation, we can only argue that any degree in $(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ splits for all A.

Further Results: Splitting and Branching

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For all noncomputable c.e. sets A:

(i) Every degree in $(\mathrm{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A)), \leq)$ splits.
(By cl-wtt upward interpolation; see last slide.)

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(iv) Open: Is every degree in $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$ branching?

Further Results: Local vs. Global Joins and Meets

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For $\mathbf{a}, \mathbf{b} \in (\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$:

- The global join $\mathbf{a} \vee \mathbf{b}$ is the least upper bound of **a** in **b** in $(\mathrm{R}_{\mathrm{ibT}}, \leq)$.
- The local join $\mathbf{a} \vee_{local} \mathbf{b}$ is the least upper bound of **a** in **b** in $(\mathrm{R}_{\mathrm{ibT}}(\mathit{deg}_{\mathrm{cl}}(A)), \leq)$(etc.).

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(ii) $\mathbf{a} \wedge \mathbf{b} \downarrow \Rightarrow \mathbf{a} \wedge_{local} \mathbf{b} \downarrow$ and $\mathbf{a} \wedge \mathbf{b} \downarrow \Rightarrow \mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge_{local} \mathbf{b}$

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BUT for some A there are $\mathbf{a}, \mathbf{b} \in (\mathrm{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq)$ such that $\mathbf{a} \vee_{local} \mathbf{b} \downarrow$ and $\mathbf{a} \vee \mathbf{b} \uparrow$ (the corresponding question for meets is still open).

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Further Research Directions and Open Problems

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- Are there c.e. sets A_n, $n \geq 0$, such that

$$\mathrm{Th}(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A_n)), \leq) \neq \mathrm{Th}(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A_{n'})), \leq)$$

$$(\ \mathrm{Th}(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A_n)), \leq) \neq \mathrm{Th}(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(A_{n'})), \leq)\ )$$

for $n \neq n'$?

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- Is the following true?

$$\forall\, A, B\ (\ \mathrm{Th}(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq) \neq \mathrm{Th}(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(B)), \leq)\ )$$

  We know that

$$\exists\, B\ \forall\, A\ (\ \mathrm{Th}(\mathbf{R}_{\mathrm{ibT}}(deg_{\mathrm{cl}}(A)), \leq) \neq \mathrm{Th}(\mathbf{R}_{\mathrm{cl}}(deg_{\mathrm{wtt}}(B)), \leq)\ )$$

THANK YOU !