The Computably Enumerable Sets: a Partial Survey with Questions

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The Computably Enumerable Sets, $\mathcal{E}$

- W_e is the domain of the eth Turing machine.
- The structure $\mathcal{E}$ is $(\{W_e : e \in \omega\}, \subseteq)$, the c.e. (r.e.) sets under inclusion.

Definability in $\mathcal{E}$:

- A set A is *computable* iff A and $\overline{A}$ are both computable enumerable.
- A set A is *finite* iff every subset of A is computable.
- $0, \omega, \cup$, $\cap$, and $\sqcup$ (disjoint union) are definable from $\subseteq$ in $\mathcal{E}$.

Definable Sets

- A set A is *simple* iff for every (c.e.) set B if $A \cap B$ is empty then B is finite.
- A set M is *maximal* iff for every (c.e.) set B either there is a finite set F such that $B \subseteq M \cup F$ or $\omega \subseteq M \cup B \cup F$.

Soare's 1974 Result

Theorem (Soare)

The maximal sets form an orbit of $\mathcal{E}$.

Definition

X is *automorphic* to Y, $X \approx Y$, iff there is an automorphism of $\mathcal{E}$ such that $\Phi(X) = Y$.

Is every orbit equally "nice" as the maximal sets?

The Scott Rank of $\mathcal{E}$ is $\omega_1^{CK}+1$

Theorem (Cholak, Downey, Harrington)

There is an c.e. set A such that the set

$$\mathcal{I}_A = \{i : A \text{ is automorphic to } W_i\}$$

is Σ^1_1-complete.

Corollary

The complexity of the $\mathcal{L}_{\omega_1,\omega}$ formula describing the above orbit is as high as possible.

Orbits of Simple Sets

Let Φ be an automorphism of $\mathcal{E}$ and f be such that $\Phi(W_e) = W_{f(e)}$. We use f to describe Φ. So Φ is effective iff f is also.

Theorem (Cholak and Harrington)

Two simple sets are automorphic iff they are Δ^0_6 automorphic.

Theorem (Harrington 2012)

The complexity of the $\mathcal{L}_{\omega_1,\omega}$ formula describing the orbit of any simple set is very low.

More Orbits

Theorem

For all $n \geq 8$, there is a properly Δ^0_n orbit (an orbit which is an orbit under Δ^0_n automorphisms).

Conjecture

We can build the above orbits to have arbitrary complexity in terms of the $\mathcal{L}_{\omega_1,\omega}$ formula describing the orbit.

Turing Complete Sets

Definition

A set c.e. A is *Turing complete* iff for every c.e. set B, B is Turing reducible to A, $B \leq_T A$.

Question (Completeness)

Which c.e. sets are automorphic to complete sets?

The orbits in the previous results all contain complete sets.

Question

Is the above question an arithmetical question?

Prompt Sets

Definition

A is *promptly simple* iff there is computable function p such that for all e, if W_e is infinite then there is a x and s with $x \in (W_{e,at\ s} \cap A_{p(s)})$.

Definition

A c.e. set A is *prompt* iff there is computable function p such that for all e, if W_e is infinite then there is a x and s with $x \in W_{e,at\ s}$ and $A_s \upharpoonright x \neq A_{p(s)} \upharpoonright x$.

Theorem (Cholak, Downey, Stob)

All prompt simple sets are automorphic to a complete set.

Almost Prompt Sets

Definition

$X = (W_{e_1} - W_{e_2}) \cup (W_{e_3} - W_{e_4}) \cup \ldots (W_{e_{2n-1}} - W_{e_{2n}})$ iff X is $2n$-c.e. and X is $2n+1$-c.e. iff $X = Y \cup W_e$, where Y is $2n$-c.e.

Definition

Let X_e^n be the eth n-c.e. set. A is *almost prompt* iff there is a computable nondecreasing function $p(s)$ such that for all e and n if $X_e^n = \overline{A}$ then $(\exists x)(\exists s)[x \in X_{e,s}^n$ and $x \in A_{p(s)}]$.

Lemma

Prompt implies almost prompt. So every Turing complete set is almost prompt.

Theorem (Harrington, Soare)

All almost prompt sets are automorphic to a complete set.

Tardy Sets

Definition

D is 2-*tardy* iff for every computable nondecreasing function $p(s)$ there is an e such that $X_e^2 = \overline{D}$ and $(\forall x)(\forall s)[\text{if } x \in X_{e,s}^2 \text{ then } x \notin D_{p(s)}]$

Theorem (Harrington, Soare)

There are realizable $\mathcal{E}$ definable properties $Q(D)$ and $P(D, C)$ such that

- *$Q(D)$ implies that D is 2-tardy (so not Turing complete),*
- *if there is a C such that $P(D, C)$ and D is 2-tardy then $Q(D)$ (and D is high),*

n-Tardy Sets

Definition

D is *n-tardy* iff for every computable nondecreasing function $p(s)$ there is an e such that $X_e^n = \overline{D}$ and $(\forall x)(\forall s)[\text{if } x \in X_{e,s}^n \text{ then } x \notin D_{p(s)}]$.

Theorem

There are realizable $\mathcal{E}$ definable properties $Q_n(D)$ such that $Q_n(D)$ implies that D is properly n-tardy (so not Turing complete).

Question

If D is not automorphic to a complete set must D satisfy some Q_n?

Degrees of n-Tardy sets

Splits of 2-tardys are 3-tardy but

Theorem (Cholak, Gerdes, Lange)

There is a 3-tardy that is not computable in any 2-tardy.

Question

Is there an $n+1$-tardy set that is not computed by any n-tardy set? Is there a very tardy sets which is not computed by any n-tardy?

Questions about Tardiness

Question

How do the following sets of degrees compare:

- *the high$_n$ hemimaximal degrees,*
- *the tardy degrees,*
- *for each n, $\{\mathbf{d}$: there is a n-tardy D such that $\mathbf{d} \leq_T D\}$,*
- *$\{\mathbf{d}$: there is a 2-tardy D such that $Q(D)$ and $\mathbf{d} \leq_T D\}$,*
- *$\{\mathbf{d}$: there is a $A \in \mathbf{d}$ which is not automorphic to a complete set$\}$.*

Theorem (Harrington, Soare)

There is a maximal 2-tardy set.

Question

Is there a nonhigh 2-tardy set which is automorphic to a complete set?

$\mathcal{D}$-Maximal Sets

Definition (The sets disjoint from A)

$\mathcal{D}(A) = \{B : \exists W (B \subseteq A \cup W \text{ and } W \cap A =^* \emptyset)\}$ under inclusion. Let $\mathcal{E}_{\mathcal{D}(A)}$ be $\mathcal{E}$ modulo $\mathcal{D}(A)$.

Definition

A is $\mathcal{D}$-*hhsimple* iff $\mathcal{E}_{\mathcal{D}(A)}$ is a Σ^0_3 Boolean algebra. A is $\mathcal{D}$-*maximal* iff $\mathcal{E}_{\mathcal{D}(A)}$ is the trivial Boolean algebra iff for all c.e. sets B there is a c.e. set D disjoint from A such that either $B \subset A \cup D$ or $B \cup D \cup A = \omega$.

Lemma

Maximal sets are $\mathcal{D}$-maximal. Plus there are many of examples of $\mathcal{D}$-maximal sets.

Question ($\mathcal{D}$-Maximal Completeness)

Does the orbit of every $\mathcal{D}$-maximal set contain a complete set?

$\mathcal{D}$-hhsimple and Simple

Theorem (Maass 84)

If A is $\mathcal{D}$-hhsimple and simple (i.e., hhsimple) if $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\hat{A})}$ then $A \approx \hat{A}$.

Theorem (Cholak, Harrington)

If A is hhsimple then $A \approx \hat{A}$ iff $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\hat{A})}$.

All such orbits contain complete sets.

Complexity Restrictions

Theorem (Cholak, Harrington)

If A is $\mathcal{D}$-hhsimple and A and $\hat{A}$ are in the same orbit then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\hat{A})}$.

Does not provide an answer to the following:

Question ($\mathcal{D}$-maximal Completeness)

Which $\mathcal{D}$-maximal sets are automorphic to complete sets?

Question

Is the above question an arithmetical question?

Question

Can we classify the $\mathcal{D}$-maximal sets?

Beginning the Classification $\mathcal{D}$-maximal sets

Definition

For a $\mathcal{D}$-maximal set, a list of c.e. sets $\{X_i\}_{i\in\omega}$ *generates* $\mathcal{D}(A)$ iff for all D if D is disjoint from A then there is a n such that $D \subseteq^* \bigcup_{i\leq n} X_i$. (This list need not be computable.)

Lemma

- *$\{\emptyset\}$ generates $\mathcal{D}(A)$ iff A is maximal.*
- *For any computable set R, $\{R\}$ generates $\mathcal{D}(A)$ iff A is maximal on $\overline{R}$.*
- *For any noncomputable c.e. set W, $\{W\}$ generates $\mathcal{D}(A)$ iff A is a trivial split of a maximal set.*
- *In all other cases, the list of generators is infinite.*

Disjoint Generators

Lemma

If infinitely many computable sets are used to (partially) generate $\mathcal{D}(A)$ we can assume that (partial) list is pairwise disjoint.

Lemma (Cholak, Gerdes, Lange)

Assume an infinite pairwise disjoint list generates $\mathcal{D}(A)$. Then either

- *All the generators are computable.*
- *All but one of the generators is computable.*
- *None of the generators are computable.*

and, in all these cases, A is automorphic to a complete set.

Other Generators

Theorem (Cholak, Gerdes, Lange)

There are four more cases of possible generators and all these cases break up into infinitely many orbits.

It is unknown if any of these orbits contain complete sets.

Nested Noncomputable Generators and Disjoint Computable Generators

Theorem

- *There are $\mathcal{D}$-maximal sets which are generated by noncomputable sets $\{W_i\}_{i\in\omega}$ and pairwise disjoint computable sets $\{R_i\}_{i\in\omega}$ such that for all $i \geq j$, $W_i \cap \mathcal{R}_j \neq^* \emptyset$ and $W_i \cap \overline{R_i} \subset^* W_{i+1}$.*
- *The class of such $\mathcal{D}$-maximal sets breaks into infinitely many orbits.*

Which sets are automorphic to low sets?

Theorem (Epstein)

There is a properly low$_2$ degree $\mathbf{d}$ *such that if* $A \leq_T \mathbf{d}$ *then* A *is automorphic to a low set.*

Definition (Following Maass)

A has the (Δ^0_3) *low shrinking property* iff for any (Δ^0_3) simultaneous enumeration of the c.e. sets $\{U_e | e \in \omega\}$ we can effectively (Δ^0_3) assign a shrinking U^S_e to each U_e such that $U^S_e \cap \overline{A} =^* U_e \cap \overline{A}$ and for finite F if $\bigcap_{e \in F} U^S_e \cap A$ is infinite then $\bigcap_{e \in F} U_e \cap A$ is infinite (entry states w.r.t. the shrunken sets are the same as the entry w.r.t. given enumeration).

Conjecture (Cholak and Weber)

A is Δ^0_3 automorphic to a low set iff A has the Δ^0_3 low shrinking property.