Strong jump-traceability 2

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SJT and randomness

Random and c.e. sets

We focus on c.e. sets in light of the inherent enumerability of strong jump-traceability.

Theorem (Diamondstone,Greenberg,Turetsky)

Every SJT set is computable from a c.e. SJT set.

Random and c.e. sets

Thesis: c.e. sets and random sets have very little in common.

- An incomplete c.e. set cannot compute a random set [Arslanov].
- A sufficiently random set cannot compute a (incomputable) c.e. set [Hirschfeldt,Miller for exact bounds].

But not all is lost.

Kučera's programme

Theorem (Kučera)

Every Δ^0_2 Martin-Löf random set computes an incomputable c.e. set.

Q: what kind of random sets compute what kind of c.e. sets?

Theorem (Hirschfeldt,Nies,Stephan)

Every c.e. set computable from an incomplete ML-random set is K-trivial.

The converse is open (but wait for André's talk).

The covering problem for SJT

Theorem (Kučera,Nies;Greenberg,Turetsky)

A c.e. degree is SJT if and only if it is computable from a Demuth random set.

The difference between Martin-Löf randomness and Demuth randomness is that when specifying components of the tests, we can change our minds a computably bounded number of times.

Lowness and randomness

We get a duality between a hierarchy of lowness notions of c.e. sets on the one hand, and of randomness (between Martin-Löf and weak 2-randomness) on the other.

- K-triviality corresponds to “Oberwolfach randomness” [Bienvenu,Greenberg,Kučera,Nies,Turetsky].
- Strong jump-traceability corresponds to Demuth randomness.

And the theme is that the strength of randomness is determined by what kind of approximations to the components of tests we allow.

What *many* random sets can compute

Theorem (Greenberg,Hirschfeldt,Nies;DGT)

The following are equivalent for a Turing degree **a***:*

- **a** *is computable from every ω-computably-approximable ML-random set;*
- **a** *is computable from every superlow ML-random set;*
- **a** *is SJT.*

Theorem (Greenberg,Hirschfeldt,Nies)

The following are equivalent for a c.e. degree **a***:*

- **a** *is computable from every superhigh ML-random set.*
- **a** *is SJT.*

So paradoxically, in the context of randomness, both superlowness and superhighness are notions of strength.

If A is SJT, then it is computable from every superlow random set.

Let Y be a superlow random set.

- Start following Kučera. The use of the planned reduction $A \leqslant_T Y$ begin with the identity. If $Y \upharpoonright_k$ changes at stage s, declare that $Y \upharpoonright_k$ should computes $A \upharpoonright_s$. (Modifying this, we see that Y can compute A with "tiny use", [Franklin,Greenberg,Stephan,Wu].)
- We use little boxes to verify that $\alpha = A_s \upharpoonright_n$ is an initial segment of A. The "weight" of α is $2^{-|\gamma|}$, where $\gamma \prec Y_s$ is currently used for computing α from Y. Use k boxes for strings of weight 2^{-k}.
- If $\alpha \nprec A$, then we will have to enumerate γ into a Solovay test. We need the total weight to be finite.

If A is SJT, then it is computable from every superlow random set.

- Use metaboxes (one for each possible weight), to ensure that we believe an erroneous string with weight 2^{-k} at most k times. Luckily, $\sum k2^{-k} < \infty$.
- When $Y \upharpoonright_k$ changes, we need to test a longer initial segment of A, and so need to run a new test - cannot use older boxes.
  If Y is superlow then we can tell in advance how many parallel tests we may need at each weight, and so how many boxes we need for each metabox.
- The reason for defining the use as we did is so that the k^{th} agent is only responsible for the latest version of $Y \upharpoonright_k$. The previous ones are passed over to $k-1, k-2, \dots$.

If A is computable from every superlow random set, then A is SJT.

A "phantom golden run construction": we construct a random set which does not exist.

Suppose A is computable from every superlow random set. We want to trace J^A.

- Start with a Π^0_1 class $\mathcal{P}_0$ of randoms, and in the background, run the argument for the (super)low basis theorem: we get a sequence $\mathcal{P}_0 = \mathcal{Q}_0, \mathcal{Q}_1, \dots$, with $\mathcal{Q}_i$ deciding the jump on the i^{th} bit.

If A is computable from every superlow random set, then A is SJT.

- From this sequence, try to generate a trace for J^A. Given n and a possible computation $J^A(n)$, with use $\alpha \prec A$, pick some i (which depends on the required bound for the trace we are enumerating). Wait until $\alpha \prec \Phi_0(X)$ for every $X \in \mathcal{Q}_i$, then believe. We will believe at most 2^i values (the number of possible versions of $\mathcal{Q}_i$), hence the bound on the trace.
- While we wait, define $\mathcal{P}_1$ to be the class of $X \in \mathcal{Q}_i$ such that $\alpha \not\prec \Phi_0(X)$. Restart the process with $\mathcal{P}_1$ and Φ_1. Cancel when A changes.
- If no level gives us a trace we let $\{Z\} = \bigcap \mathcal{P}_n$. Then Z does not compute A. We string together the superlow basis construction to show that Z is superlow.

SJT and the c.e. degrees

First application: superlow cupping

Theorem (Greenberg,Nies;DGT)

Every SJT degree $\mathbf{a}$ *is superlow preserving: for every superlow degree* $\mathbf{b}$*,* $\mathbf{a} \vee \mathbf{b}$ *is also superlow.*

Corollary (Diamondstone)

The notions of low cupping and superlow cupping differ in the c.e. degrees.

Relativising SJT

Lowness notions can often be partially relativised to obtain “weak reducibilities”. For example, K-triviality leads to $\leqslant_{\mathrm{LR}}$, a relation which measures how well an oracle derandomises ML-random sets.

Definition (Nies)

Let $A, B \in 2^\omega$. Then $A \leqslant_{\mathrm{SJT}} B$ if for every order function h, every A-partial computable function has a B-c.e. h-trace.

Question

Does $\leqslant_{\mathrm{SJT}}$ imply $\leqslant_{\mathrm{LR}}$?

SJT-hard degrees

Definition

- A set A is LR-hard if $\emptyset' \leqslant_{\mathrm{LR}} A$.
- A set A is SJT-hard if $\emptyset' \leqslant_{\mathrm{SJT}} A$.

Theorem (Kjos-Hanssen,Miller,Solomon)

A Turing degree is LR*-hard if and only if it is almost everywhere dominating.*

Question (Nies,Shore,...)

In the c.e. degrees, is there a minimal pair of LR*-hard degrees?*

Pseudojump operators

There are direct constructions of incomplete LR-hard and SJT-hard c.e. degrees. An indirect approach uses pseudojump inversion.

Definition (Jockusch,Shore)

A pseudojump operator is a function $J\colon 2^\omega \to 2^\omega$ such that for all $A \in 2^\omega$, $J(A)$ is uniformly c.e. in A and uniformly computes A. A pseudojump operator is increasing if for all A, $J(A) >_T A$.

Theorem (Jockusch,Shore)

For any pseudojump operator J there is a c.e. set A such that $J(A) \equiv_T \emptyset'$.

Question (Jockusch,Shore)

Can this be combined with upper-cone avoidance? Can one always invert to minimal pairs?

Partial answers by Downey,Jockusch,LaForte.

Restrictions on pseudojump inversion

Theorem (Downey,Greenberg)

There is no minimal pair of SJT*-hard c.e. degrees. In fact, there is an incomputable c.e. set which is computable in every* SJT*-hard c.e. set.*

Corollary

There is a natural, increasing pseudojump operator J_{SJT} *which cannot be inverted to a minimal pair, or while avoiding upper cones.*

No minimal pair

This is an "inverted" box-promotion argument. Suppose that both A_0 and A_1 are c.e. and SJT-hard. We want to build an incomputable c.e. set E below both A_0 and A_1.

- Friedberg-Muchnik actors will want to put a numbers into E. Such enumerations will require simultaneous permission from both A_0 and A_1.
- We can encourage A_i to change by changing the values of a partial $\Sigma^0_2 = \Sigma^0_1(\emptyset')$ function ψ and waiting for A_i to enumerate the current value in its trace T^{A_i} for ψ.
- If z is a 1-box – the bound on the trace is 1 – then every change in $\psi(z)$ forces a change in A_i below the use of enumerating the current value $\psi(z)$ in $T^{A_i}(z)$. We tie uses together so that this change permits a "follower" into E.

No minimal pair

- If z is a 2-box – we may need to ask twice before we get a change.
- But how do we get simultaneous change in A_0 and A_1? The only way is if we have 1-boxes on both sides.
- Boxes can be tied up by followers waiting to be realised. So the supply is limited.
- Box promotion is used to eventually manufacture 1-boxes from larger boxes. Again large metaboxes are used so that the gains from each promotion can be distributed to many mouths. See zig-zag picture.

LR-hard minimal pair

Recall the question, “is there a minimal pair of c.e., LR-hard degrees?”.

A solution may be found using the same technique.

- If every K-trivial degree is $\frac{1}{10}\log(n)$-jump traceable, then there is no minimal pair of LR-hard degrees.

(Recall that every K-trivial degree is $M \log n$-jump traceable for some M, but some K-trivial degree is not $o(\log n)$-jump traceable. So this is related to the problem of finding a combinatorial characterisation for K-triviality.)

The ideal SJTH♠ of all c.e. degrees which are reducible to all SJT-hard c.e. degrees is a new ideal in the c.e. degrees.

The extent of this ideal measures how restricted the construction of an incomplete SJT-hard c.e. set is.

Maximality

Question

Is the ideal $\mathrm{SJTH}^{\spadesuit}$ *principal?*

This question is difficult because the usual way for showing an ideal is not principal is by using... lower-cone avoidance.

An attempt at an answer

Theorem (Diamondstone,Downey,Greenberg,Turetsky)
$\mathrm{SJTH}^{\spadesuit}$ *contains a superhigh set, but no* SJT*-hard set.*

One hope is to use the superhighness hierarchy to obtain an answer.

Thank you