Diagonal noncomputability and randomness

(coverability, cuppability)

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The Incomputable

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Remark

- It is a survey of properties of PA sets, DNC functions and $\mathcal{C}$-random sets (for various randomness notions $\mathcal{C}$)
- Results of various authors
- A special focus: an information content of these objects (strength and weakness)

PART I

Classes PA and DNC

Definition

- A total function f is diagonally noncomputable (DNC) if $f(e) \neq \varphi_e(e)$ for all e
- DNC functions from 2^ω, i.e. $(0,1)$-valued, are called PA sets. The class of all PA sets is denoted by $\mathcal{PA}$.

Definition (Simpson)

$\mathbf{b} << \mathbf{a}$ means that every infinite tree $T \subseteq 2^{<\omega}$ of degree $\leq \mathbf{b}$ has an infinite path of degree $\leq \mathbf{a}$.

Theorem (D. Scott, Solovay and others)

The following conditions are equivalent:

1. $\mathbf{a}$ *is a degree of a* $(0,1)$*-DNC function*
2. $\mathbf{a} >> \mathbf{0}$
3. $\mathbf{a}$ *is a degree of a complete extension of PA*
4. $\mathbf{a}$ *is a degree of a set separating some effectively inseparable pair of c.e. sets.*

Remark

1. $\mathcal{PA}$ is a kind of a “universal” Π^0_1 class
2. degrees $>> \mathbf{0}$ are called PA degrees
3. (Simpson) The partial ordering $<<$ is dense
4. There are low PA sets A, i.e., $A' \equiv_T \emptyset'$ (by Low Basis Theorem of Jockusch, Soare)

Since Σ^0_1 conditions true in $\mathbb{N}$ are provable in PA, any c.e. set has to be coded in some complete extension of PA.

Question: Which properties of a c.e. set (or a family of c.e. sets) can be guaranteed by which (= how complicated) PA sets?

Theorem (K)

For any Δ^0_2 PA set X (or Δ^0_2 DNC function) there is a noncomputable c.e. set $A <_T X$.

Taking a low PA set X, incompleteness of a resulting c.e. set A is guaranteed by incompleteness of X.
Other examples: avoiding an upper cone, splitting c.e. sets, etc.

But, Δ^0_2 PA sets cannot guarantee minimality of a pair of c.e. sets.

Theorem (K)

For any two Δ^0_2 PA sets (or Δ^0_2 DNC functions) there is a noncomputable c.e. set T-below both of them.

Question: An example:

Given an incomplete c.e. set A is there always a PA set X such that $A <_T X <_T \emptyset'$? (Can incompleteness of A be guaranteed by an incomplete PA set?)

Answer:

- No
- Only for low_1 c.e. sets.

First partial answer:

Theorem (Slaman,K)

There is an incomplete c.e. A joining to $\emptyset'$ all Δ^0_2 DNC functions

Further progress

Case PA:

Theorem (Day, Reimann)

Low_1 c.e. degrees are exactly those c.e. degrees $\mathbf{a}$ for which there is a PA degree $\mathbf{e}$ such that $\mathbf{a} < \mathbf{e} < \mathbf{0}'$ (thus, any non-low_1 c.e. degree joins to $\mathbf{0}'$ all Δ^0_2 PA degrees).

Lemma (Day, Reimann)

Given a PA set X and a c.e. set A, $X \not\geq_T A$ implies $X \oplus A \geq_T \emptyset'$.

Day and Reimann used an indirect and complicated method based on neutral measures (due to Day, Miller).

There is a short and straightforward proof using only c.e.-ness and PA-ness.

Proof (K)

Idea:
Given: A a c.e. set, X a complete extension of PA. If $X \not\geq_T A$, then X cannot correctly code A and A has to be blown in X. It can be used for coding $\emptyset'$ into $X \oplus A$.

An advantage of the straightforward proof: a stronger statement.

Theorem (K)

Given a PA set X and a c.e. set A, $X \not\geq_m A$ implies $X \oplus A \geq_T \emptyset'$.

Corollary (K)

For any PA set X and a c.e. set A at least one of the following conditions hold:

- $X \oplus A \geq_T \emptyset'$
- $X \geq_m A$ *and* X *is PA relatively in* A

Case: All Δ^0_2 DNC and superhigh c.e.

Theorem (Bienvenu, Greenberg,, Nies, Turetsky, K)

All superhigh c.e. sets join to $\emptyset'$ all Δ^0_2 DNC functions.

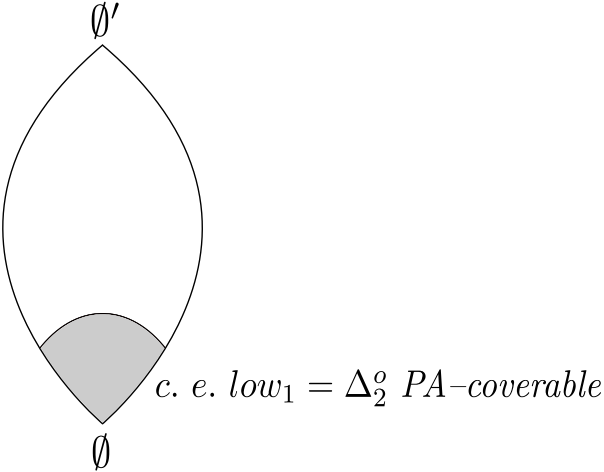
Remark

It can be proved either by using Kolmogorov complexity (relative to $\emptyset'$), or straightforwardly using just c.e.-ness and DNC-ness.

Case: All Δ^0_2 DNC and c.e. neither superhigh nor low_1

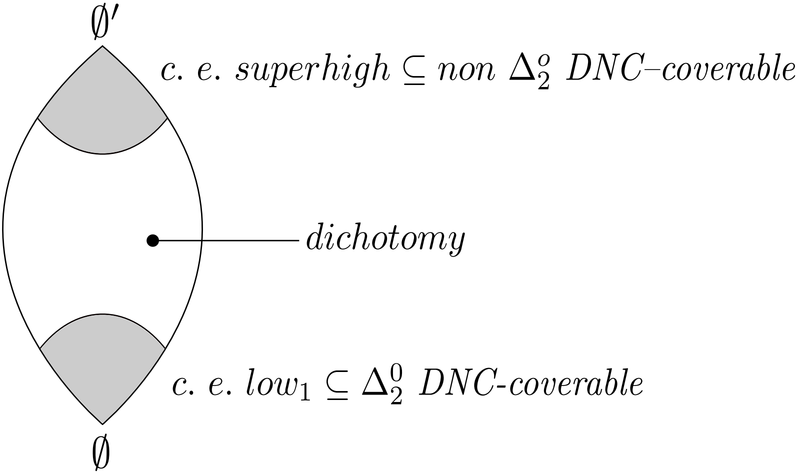
For c.e. sets which are neither low_1 nor superhigh there is a dichotomy. Some do join to $\emptyset'$ all Δ^0_2 DNC functions, some do not (for c.e. $\mathbf{a}$ the latter means: $\mathbf{a} < \mathbf{e} < \mathbf{0}'$ for some DNC degree $\mathbf{e}$) (Slaman, K).

Case: Δ^0_2 PA-coverability for c.e.:
(for c.e.:) joining to $\emptyset'$ all Δ^0_2 PA = non-Δ^0_2 PA-coverability



Case: Δ^0_2 DNC-coverability for c.e.:

(for c.e.:) joining to $\emptyset'$ all Δ^0_2 DNC = non-Δ^0_2 DNC-coverability



PART II

$\mathcal{C}$ -Randomness

$\mathcal{C}=$ ML, Dem, wDem, balanced, difference, OW, ...

A calibration of a concept of null classes - a lot of variants

Definition (Martin-Löf Randomness)

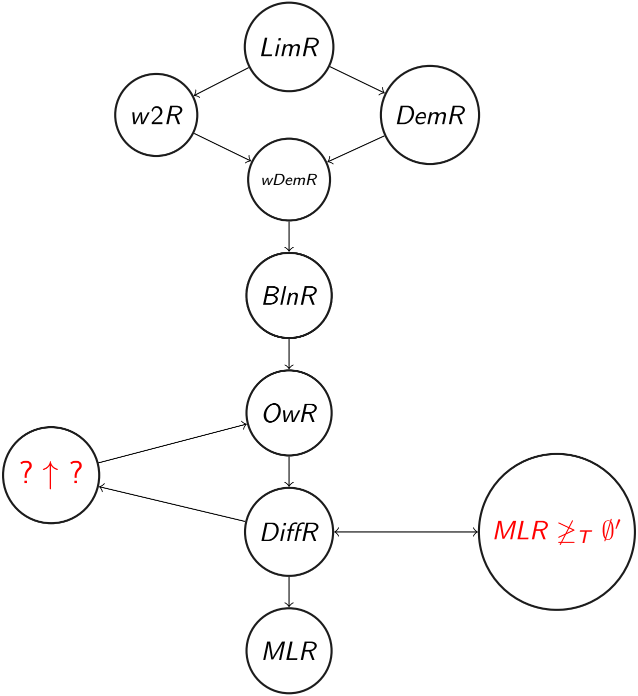
A ML-test is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of c.e. open sets such that $\forall n\ \mu(G_m) \leq 2^{-m}$.
A set Z fails the test if $Z \in \bigcap_{m\in\mathbb{N}} G_m$ otherwise Z passes the test.
Z is ML-random if Z passes each ML-test.

Definition

- A *Demuth test* is a sequence of c.e. open sets $(S_m)_{m\in\mathbb{N}}$ such that $\forall m\, \mu(S_m) \leq 2^{-m}$, and there is a function $f \leq_{\mathrm{wtt}} \emptyset'$ such that $S_m = [W_{f(m)}]^{\prec}$ (i.e. an index of S_m is given by an ω-c.e. function).
  A set Z passes the test if $Z \notin S_m$ for almost every m.
  Z is *Demuth random* if Z passes each Demuth test.
- A set Z is weakly Demuth random if for each Demuth test $Z \notin \bigcap_m S_m$ (we could require $S_m \supseteq S_{m+1}$ for each m)
- A balanced test is a Demuth test where $S_m = [W_{f(m)}]^{\prec}$ for some 2^m-c.e. function f (i.e. an index of S_m is given by an 2^m-c.e. function).
  A set Z is balanced random if Z passes any balanced test (i.e. $Z \notin \bigcap_m S_m$).

Definition

- A difference test is a sequence $(V_m)_{m\in\mathbb{N}}$ of uniformly c.e. open sets and a Π^0_1 set P such that $\mu(P \bigcap V_m) \leq 2^{-m}$ for every m. A set Z is difference random if for any difference test $((V_m)_{m\in\mathbb{N}}, P)$ we have $Z \notin (\bigcap_{m\in\mathbb{N}} V_m) \bigcap P$.
- A OW-test (OW = Oberwolfach) is a Demuth test $(S_m)_{m\in\mathbb{N}}$ such that $\forall m \, \mu(S_m) \leq 2^{-m}$, $S_m \supseteq S_{m+1}$, S_0 never changes and if we change S_{m+1} twice then we also have to change S_m ("change" means a change of index).
  A set Z is OW-random if it passes every OW-test (i.e. $Z \notin \bigcap_m S_m$).


LimR
w2R
DemR
wDemR
BlnR
OwR
? ↑ ?
DiffR
MLR ≱_T ∅′
MLR

Posner, Robinson:
Any noncomputable Δ^0_2 set can be joined to $\emptyset'$ by a 1-generic set.

Question: Moving from Baire caregory to measure ?

- ML-cupping: Can any noncomputable Δ^0_2 set A be joined up to $\emptyset'$ by an incomplete ML-random set X ?
- ML-covering: If A is a c.e. K-trivial, is there a ML-random $Z \geq_T A$ such that $\emptyset' \not\leq_T Z$?

A notion of computational weakness:

Definition

A set A is K-trivial if $K(A \upharpoonright n) \leq K(0^n) + O(1)$, for all n.

Theorem (Nies; Hirschfeldt,Nies; Hirschfeldt,Nies,Stephan)

The following conditions are equivalent for any set A

- *A is K-trivial*
- *A low for K, (i.e. $K^A =^+ K$)*
- *A low for random (i.e. ML-rand^A = ML-rand)*
- *A is a basis for ML-randomness (i.e. there is $Z, A \leq_T Z$ such that $Z \in$ ML-rand^A)*

ML-cupping:

Conjecture (K)

K-trivials can be characterized as sets which cannot be joined to $\emptyset'$ by incomplete ML-randoms.

Theorem (Day, Miller)

- *K-trivials are exactly those sets which cannot be joined to (or even above) $\emptyset'$ by an incomplete ML-random*
- *Δ^0_2 sets which are not K-trivial can be joined to $\emptyset'$ by a low ML-random set.*

Proof (Day, Miller) uses, among others, the following fact:

(Bienvenu, Hölzl, Miller, Nies): incomplete ML-randoms = ML-randoms with a positive lower Lebesgue density in every Π^0_1 class to which they belong.

ML-covering:

Definition
A is ML-coverable if for some incomplete ML-random Z, $A \leq_T Z$ (similarly $\mathcal{C}$-coverable for $\mathcal{C} =$ ML, OW, Demuth etc.).

Old theorem (Gacs, K): For any A there is always a complete ML-random Z, $A \leq_T Z$.

(Hirschfeldt, Nies, Stephan): All ML-coverable c.e. sets are K-trivial.

Big Open Question Are all K-trivials ML-coverable?

Partial progress - limitations.

(Nies, K): There is a c.e. K-trivial which is not weakly Demuth coverable.

ML-covering:

The best at the moment:

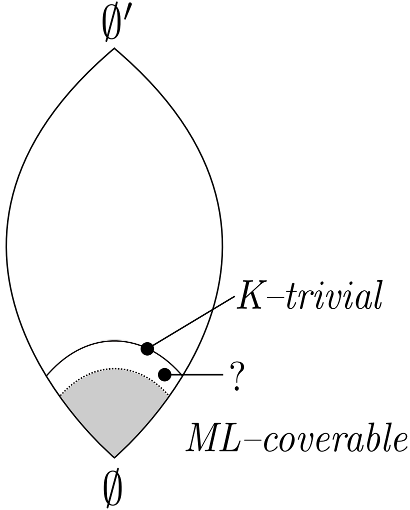
(Bienvenu, Greenberg, Nies, Turetsky, K):

- There is a c.e. K-trivial set which is not OW-coverable, i.e., no $Z \geq_T A$ is OW-random.
- Any ML-random set which is not OW-random is LR-hard and, thus, superhigh.
- If A is K-trivial then A is computable from any ML-random set which is not OW-random.

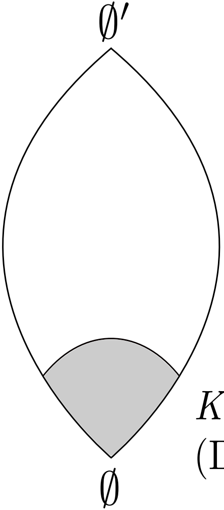
Corollary

A c.e. set not OW-coverable (as above) is not computable from (both) two halves of any ML-random set.

Δ^0_2 ML-coverability for c.e. :

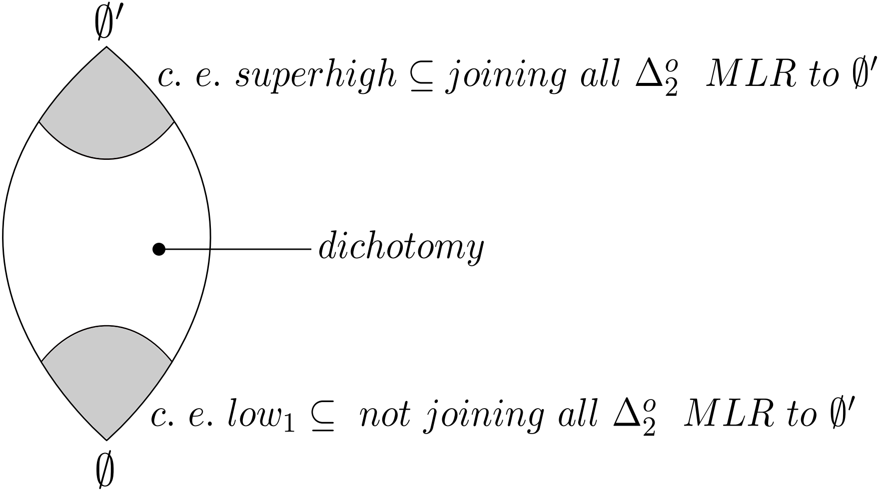


C.e. cuppable to $\emptyset'$ by an incomplete Δ^0_2 MLR:



K–trivial = non ML–cuppable
(Day, Miller)

C.e. joining to $\emptyset'$ all Δ^0_2 ML-random:



Usefulness of Demuth randomness

Definition

- An order function: a computable, non-decreasing, unbounded function (with positive values)
- A c.e. trace with bound h is a uniformly c.e. sequence $(T_x)_{x \in \mathbb{N}}$ of finite sets such that $|T_x| \leq h(x)$ for each x
- A set A is jump traceable if for some order h there is a c.e. trace $(T_x)_{x \in \mathbb{N}}$ with bound h such that for all $x \in \mathrm{dom}(\psi)$, $\psi(x) \in T_x$ for every partial function computable in A
- A set A is strongly jump traceable if it is jump traceable with bound h for each order function h

Theorem (Nies, K $\{\rightarrow\}$; Greenberg, Turetsky $\{\leftarrow\}$)

Demuth coverable c.e. sets are exactly c.e. sets which are SJT.

Thank you