Alan Turing in the Twenty-first Century:
Normal Numbers, Randomness, and Finite Automata

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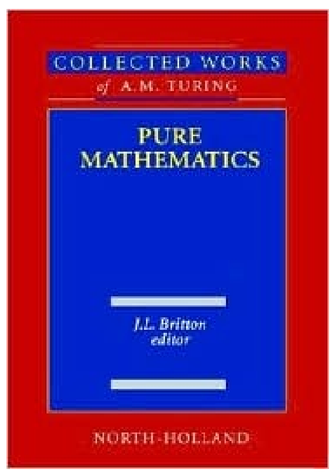
Main reference

A note on normal numbers

in



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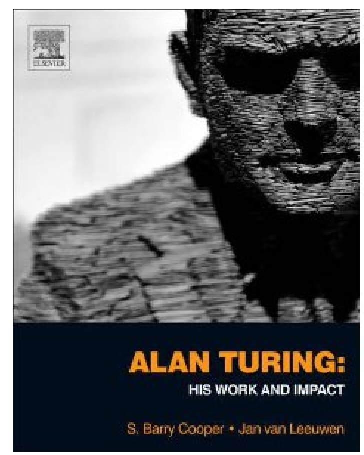
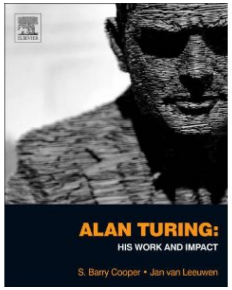


Main references on main reference

- V. Becher, Turing's note on normal numbers, in



- V. Becher, S. Figueira, and R. Picchi, Turing's unpublished algorithm for normal numbers, *Theoretical Computer Science* (2007).

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Today’s topic: Why we should care.

Outline

1. Normal Numbers
2. Explicit Constructions
3. Normality and Finite Automata
4. Conclusion

Normal Numbers

Definition (Borel 1909). Let $\alpha \epsilon \mathbb{R}$ and $2 \leq b \epsilon \mathbb{N}$.

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1. α is normal in base b if, for every $m \geq 1$ and every $w \in \{0, \ldots, b-1\}^m$, the asymptotic, empirical frequency of w in the base-b expansion of α is b^{-m}.

2. α is absolutely normal if it is normal in every base $b \geq 2$.

Theorem (Borel 1909). Almost every real number is absolutely normal.

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Answer (Cassels 1959, Schmidt 1960). Yes!

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But no natural example has been proven to be normal, even in a single base!

(Bailey & Crandall 2001 proposed a dynamical hypothesis that implies the base-2 normality of π, $\sqrt{2}, \ln 2\,, \zeta(3)$.)

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This and similar ideas work in any base …

… but not in all bases. How do we explicitly construct absolutely normal numbers?

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But let's look at that second sentence.

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This is the visionary content of Turing’s note!

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… thereby yielding constructions of computable real numbers that are absolutely normal.

Turing’s Vision

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This vision is crucial for present-day investigations of

individual random sequences,
dimensions of individual sequences,
measure and category in complexity classes,

etc.

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Definition. A martingale is a function

$$d: \{0,1\}^* \to [0,\infty)$$

satisfying

$$d(w) = \frac{d(w0)+d(w1)}{2}$$

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for all $w \epsilon \{0,1\}^*$.

A martingale d succeeds on a sequence $S \epsilon \{0,1\}^\infty$ if

$$\limsup_{w \to S} d(w) = \infty\ .$$

Definition. A real $\alpha \epsilon \mathbb{R}$ is polynomial time computable, and we write $\alpha \epsilon P_{\mathbb{R}}$, if there is a function $\hat{\alpha}: \mathbb{N} \to \mathbb{Q}$ such that

(i) for all $r \epsilon \mathbb{N}$, $|\hat{\alpha}(r) - \alpha| \leq 2^{-r}$, and

(ii) $\hat{\alpha}$(r) is computable in time polynomial in r.

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Definition. A set X $\subseteq \mathbb{R}$ has measure 0 in $P_{\mathbb{R}}$, and we write $\mu(\mathrm{X}|P_{\mathbb{R}}) = 0$, if there is a polynomial time computable martingale d that succeeds on (the binary expansion of) every element of X.

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It gives a coherent notion of measure in $P_{\mathbb{R}}$. The sets X with $\mu(X|P_{\mathbb{R}}) = 0$ form a "polynomial time ideal" in $P_{\mathbb{R}}$, and this ideal is proper, i.e. $\mu(P_{\mathbb{R}}|P_{\mathbb{R}}) \neq 0$.

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The measure conservation theorem that proves $\mu(P_{\mathbb{R}}|P_{\mathbb{R}}) \neq 0$ gives an explicit construction, from any polynomial time computable martingale d, of a real $\alpha \epsilon P_{\mathbb{R}}$ on which d does not succeed.

Theorem (Strauss 1997). Almost every polynomial time computable real number is absolutely normal. That is, if X is the set of reals that are not absolutely normal, then $\mu(X|P_{\mathbb{R}}) = 0$.

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Theorem (Mayordomo 2012). Explicit constructions of absolutely normal reals α computable in $O(n \log n)$ time.

Question

The above measure is induced on $P_{\mathbb{R}}$ by the binary expansions of reals.

Can this be done (directly) in terms of other representations of reals, e.g., the overlapping intervals representation proposed in Turing's 1937 Correction?

Normal Numbers and Finite Automata

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This connection is made via finite-state compressors and finite-state gamblers.

Definition (Shannon 1948). Fix a finite alphabet Σ.

1. A finite-state compressor (FSC) is a 4-tuple

$$C = (Q, \delta, q_0, \nu),$$

where Q, δ, q_0 form a finite-state automaton, and $\nu: Q \times \Sigma \longrightarrow \{0,1\}^*$ is the output function.

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2. The output of C on input $w \epsilon \Sigma^*$ is the string $C(w) \epsilon \{0,1\}^*$ defined by

$$C(\lambda) = \lambda; \;\; C(wa) = C(w)\nu(\delta(w), a).$$

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3. An information-lossless FSC (ILFSC) is an FSC for which the function

$$w \longmapsto \big(C(w), \delta(w)\big)$$

is one-to-one.

Definition (Schnorr and Stimm 1972).

1. A finite-state gambler (FSG) is a 4-tuple

$$G = (Q, \delta, q_0, B)$$

where Q, δ, q_0 form a finite-state automaton, and $B: Q \longrightarrow \Delta_{\mathbb{Q}}(\Sigma)$ is the betting function.
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2. The martingale of G is the function

$$d_G: \Sigma^* \longrightarrow [0, \infty)$$

$$d_G(\lambda) = 1$$

$$d_G(wa) = |\Sigma| d_G(w) B\big(\delta(w)\big)(a).$$

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3. For $s \epsilon [0,\infty)$, the s-gale of G is the function

$$d_G^{(s)}(w) = 2^{(s-1)|w|} d_G(w).$$

Definition. Let d be a gale, and let $S \epsilon \Sigma^{\infty}$.

1. d succeeds on S if $\limsup_{w \to S} d(w) = \infty$.
2. d succeeds strongly on S if $\liminf_{w \to S} d(w) = \infty$.

Definition and Theorem Let $S \epsilon \Sigma^{\infty}$.

1. (Dai, Lathrop, Lutz, and Mayordomo 2004). The finite-state dimension of S is

$$\dim_{FS}(S) = \inf \left\{ s \epsilon [0,\infty) \,\middle|\, (\exists\ FSG\ G)\ d_G^{(s)} \text{ succeeds on } S \right\}$$

$$= \inf_{\substack{C \\ \text{ILFSC}}} \liminf_{w \to S} \frac{|C(w)|}{|w| \log|\Sigma|}. \quad \text{"compression ratio"}$$

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2. (Athreya, Hitchcock, Lutz, and Mayordomo 2007). The finite-state strong dimension of S is

$$\mathrm{Dim}_{FS}(S) = \inf \left\{ s \epsilon [0,\infty) \,\middle|\, (\exists\ FSG\ G)\ d_G^{(s)} \text{ succeeds strongly on } S \right\}$$

$$= \inf_{\substack{C \\ \text{ILFSC}}} \limsup_{w \to S} \frac{|C(w)|}{|w| \log|\Sigma|}.$$

Definition. For $\alpha \epsilon \mathbb{R}$ and $2 \leq b \,\epsilon\, \mathbb{N}$, the base-$b$ finite-state dimension and finite-state strong dimension of α are

$$\dim_{\mathrm{FS}}^{(b)}(\alpha) = \dim_{FS}(S),$$

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where S is "the" base-b expansion of α.

The connection between normality and finite automata:

Theorem (Schnorr and Stimm 1972; Bourke, Hitchcock, and Vinodchandran 2005). A real $\alpha \epsilon \mathbb{R}$ is normal in base b if and only if $\dim_{\mathbf{FS}}^{(b)}(\alpha) = 1$.

Question: How true is the following statement?

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Every theorem about normal numbers is the dimension-1 special case of a more general theorem about finite-state dimension.

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The known instances of this phenomenon are interesting, because the generalizations require new methods.

Instance 1: Real arithmetic

Theorem (Wall 1949). For every $\alpha \epsilon \mathbb{R}$ and $0 \neq q \epsilon \mathbb{Q}$, if α is normal in base b, then so are $q + \alpha$ and qα.

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Theorem (Wall 1949). For every $\alpha \epsilon \mathbb{R}$ and $0 \neq q \; \epsilon \; \mathbb{Q}$, if α is normal in base b, then so are $q + \alpha$ and qα.

Theorem (Doty, Lutz, and Nandakumar 2007) For every $\alpha \epsilon \mathbb{R}$ and $0 \neq q \; \epsilon \; \mathbb{Q}$,

$$\dim_{\mathrm{FS}}^{(b)}(q+\alpha) = \dim_{\mathrm{FS}}^{(b)}(q\alpha) = \dim_{\mathrm{FS}}^{(b)}(\alpha)$$

and

$$\mathrm{Dim}_{\mathrm{FS}}^{(b)}(q+\alpha) = \mathrm{Dim}_{\mathrm{FS}}^{(b)}(q\alpha) = \mathrm{Dim}_{\mathrm{FS}}^{(b)}(\alpha)\,.$$

Instance 2. Copeland-Erdös sequences

Definition. The base-b Copeland-Erdös sequence of an infinite set $A \subseteq \mathbb{Z}^+$ is

$CE_b(A) =$ the concatenation of the base$-b$ expansions of the elements of A in order.

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Champernowne also conjectured that $CE_{10}(\text{PRIMES})$ is normal in base 10.

Theorem (Copeland and Erdös 1946) If $A \subseteq \mathbb{Z}^+$ is "sufficiently dense", then $CE_b(A)$ is normal in base b. (And PRIMES is sufficiently dense by the Prime Number Theorem)

Definition. Let $A \subseteq \mathbb{Z}^+$.

1. The A-zeta function $\zeta_A \colon [0,\infty) \to [0,\infty]$ is defined by

$$\zeta_A(s) = \sum_{n \in A} n^{-s}.$$

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1. The A-zeta function $\zeta_A : [0,\infty) \to [0,\infty]$ is defined by

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2. The zeta-dimension of A is

$$\mathrm{Dim}_\zeta(A) = \inf\{s \mid \zeta_A(s) < \infty\}$$

$$= \limsup_{n \to \infty} \frac{\log|A \cap \{1, \ldots, n\}|}{\log n}.$$

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$$= \limsup_{n \to \infty} \frac{\log|A \cap \{1, \dots, n\}|}{\log n} .$$

3. The lower zeta-dimension of A is

$$\mathrm{dim}_\zeta(A) = \liminf_{n \to \infty} \frac{\log|A \cap \{1, \dots, n\}|}{\log n} .$$

Theorem, restated (Copeland and Erdös 1946). If $\dim_{\zeta}(A) = 1$, then $CE_b(A)$ is normal in base b.

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Theorem (Gu, Lutz, and Moser 2007).

$$\begin{array}{ccc} \dim_{FS}\big(CE_b(A)\big) & \geq & \dim_\zeta(A) \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ \mathrm{Dim}_{FS}\big(CE_b(A)\big) & \geq & \mathrm{Dim}_\zeta(A) \end{array}$$

and that's all.

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and that's all.

Note:

$$\dim_{FS}\big(CE_b(\text{SQUARES})\big) = 1 > \frac{1}{2} = \dim_\zeta(\text{SQUARES})$$

Besicovitch 1936; Davenport and Erdös 1952

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But Turing, hypothetically apprised of these developments, might well ask a different, more specific question.

Definition. A real number α is absolutely dimensioned if

$$\dim_{FS}(\alpha) = \dim_{FS}^{(b)}(\alpha)$$

does not depend on b.

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Is there an absolutely dimensioned real number α with

$$0 < \dim_{FS}(\alpha) < 1 \quad ?$$

If so, can we explicitly construct such numbers?

Conclusion

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Thank you!