# A computability theoretic equivalent to Vaught's conjecture.

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For the rest of the talk, all our structures are countable

# Background on infinitary logic

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**Def:** For structures $\mathcal{A}$ and $\mathcal{B}$, and $\alpha \in \omega_1$,
we write $\mathcal{A} \equiv_\alpha \mathcal{B}$ if they satisfy the same $\Pi^{in}_\alpha$-sentences.

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- For each $\alpha < \omega_1$, there are countably many $\mathcal{A} \models T$ with $\rho(\mathcal{A}) = \alpha$.
- So $|\{\text{models of } T\}| \leq \aleph_1$.

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**Note:** This definition is independent of whether CH holds or not.

# The Main Theorem–again

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($0^{(\alpha)}$ is the $\alpha$th Turing jump of 0.)

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- $X = \{n \in \omega : \varphi(n)\}$, where $\varphi$ is a computable infinitary formula.
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Obs: All arithmetic sets are hyperarithmetic.

# Hyperarithmetic-is-Recursive

Let $\mathbb{K}$ be a class of structures.

**Def:** $\mathbb{K}$ satisfies *hyperarithmetic-is-recursive* if

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**Def:** $\mathbb{K}$ satisfies *hyperarithmetic-is-recursive on a cone* if,
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**Theorem ([M.] (ZFC+PD))**

*Let $(\mathfrak{K}, \equiv, r)$ be scattered projective ranked equivalence relation such that $\forall Z \in \mathfrak{K},\ r(Z) < \omega_1^Z$.*
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**Lemma:** [Martin] (ZFC+PD) If $f \colon 2^\omega \to \omega_1$ is projective and $f(X) < \omega_1^X$,
then $f$ is constant on a cone.

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Using the Ulm rank on $p$-groups with finite dimensional divisible part.

# Scott Rank

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**Corollary:** (ZFC+PD)
If $T$ is scattered, the class of models of $T$ of *low Scott rank*
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**Obs:** For every structure $\mathcal{A}$, $\rho(\mathcal{A}) \leq \omega_1^{\mathcal{A}}$.

# The reversal

**Theorem (M.)**

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The technique:

**Lemma (M. 09)**

*If $\mathbb{K}$ has a $2^{\aleph_0}$ many $\equiv_\alpha$-equivalence classes, then, relative to an oracle, for every $X$, there is $\mathcal{A}$ in $\mathbb{K}$ which weakly codes $X$ in its $\alpha$th-jump.*

# The Main Theorem–again

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*If $T$ is a counterexample to Vaught's conjecture, then, there is $Y \in 2^\omega$ such that*

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**Corollary:** If $T$ is a counterexample to Vaught's conjecture, then $T$ satisfies hyperarithmetic-is-recursive on a cone.

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**Obs:**

- $\mathbb{K}$ is $\Sigma^1_1$.
- $\mathbb{K}$ is not $L_{\omega_1,\omega}$ axiomatizable.

# The technique

**Def:** $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ if $\Pi^{in}_\alpha$-Th$(\mathcal{A}, \bar{a}) \subseteq \Pi^{in}_\alpha$-Th$(\mathcal{B}, \bar{b})$.

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**Def:** Let $\mathbf{bf}_\alpha(\mathbb{K}) = \frac{\{(\mathcal{A},\bar{a}):\mathcal{A}\in\mathbb{K},\bar{a}\in\mathcal{A}^{<\omega}\}}{\equiv_\alpha}$, the set of $\Pi^{in}_\alpha$-types realized in $\mathbb{K}$.

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- the projections $(\cdot)_\beta \colon \mathbf{bf}_{\geq\beta} \to \mathbf{bf}_\beta$ for all $\beta < \alpha$;
- the projections $\pi_\iota(\cdot) \colon \mathbf{bf}_{\beta,k} \to \mathbf{bf}_{\beta,\ell}$ for all $\beta \leq \alpha, k, \ell \in \omega, \iota \in \{1, ..., \ell\}^k$;
- the binary relations $\{(\tau, \sigma) : \tau \in ext_\beta(\sigma)\} \subseteq \mathbf{bf}_\beta \times \mathbf{bf}_{>\beta}$ for all $\beta < \alpha$; and
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**Theorem (M.)**

*Let $\mathbb{B}$ be a presentation of the $\alpha$-bf-structure of $T$ and $\sigma \in \mathbf{bf}_{\alpha,k}$. Then $\mathbb{B}$ can compute a structure $(\mathcal{A}, \bar{a})$ of $\alpha$-bf-type $\sigma$.*

# The main theorem.

**Theorem ([M.] (ZFC+PD))**

*Let $T$ be a theory with uncountably many countable models.*
*The following are equivalent:*

- *$T$ is a counterexample to Vaught's conjecture.*
- *$T$ satisfies hyperarithmetic-is-recursive on a cone.*
- *There exists an oracle relative to which*

$$\{Sp(\mathcal{A}) : \mathcal{A} \models T\} \;=\; \{\{X \in 2^{\omega} : \omega_1^X \geq \alpha\} : \alpha \in \omega_1\}.$$