On the Complexity of Finding Narrow Proofs

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HU Berlin $\longrightarrow$ RWTH Aachen

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Finding small width resolution refutations

One heuristic for 3-CNF SAT-Solving (Galil '77): Find resolution refutations of minimal width (= size of the largest clause in the refutation).

$i \leftarrow 0$.
repeat
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- Size S treelike refutation $\Longrightarrow$ refutation width $\leq \log S$. (Ben-Sasson, Widgerson '01)

Are there small width refutations?

Resolution width problem

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- k part of input: EXPTIME-complete.

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- k fixed: $\notin$ DTIME$(n^{\frac{k-3}{12}})$.

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- k part of input: EXPTIME-complete.
- k fixed: $\notin$ DTIME$(n^{\frac{k-3}{12}})$.
- k parameter: XP-complete, hence $\notin$ FPT.

A game characterization of resolution width

The Boolean existential $k+1$ pebble game (Atserias, Dalmau '08):

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- Spoiler stores assignments of at most $k+1$ variables.
- Spoiler can delete assignments at any time.
- Spoiler wins, if the current assignment falsifies an initial clause.

Bounded width resolution

Spoiler wins the Boolean ex. $(k+1)$-pebble game on Γ
$\iff \Gamma$ has a width-k resolution refutation (Atserias, Dalmau '08)

$$\Gamma = \{D \vee \overline{E},\ C \vee D \vee E,\ A \vee B \vee \overline{C},\ \overline{A} \vee D \vee E,\ \overline{B} \vee \overline{C} \vee D,\ \overline{D}\}$$

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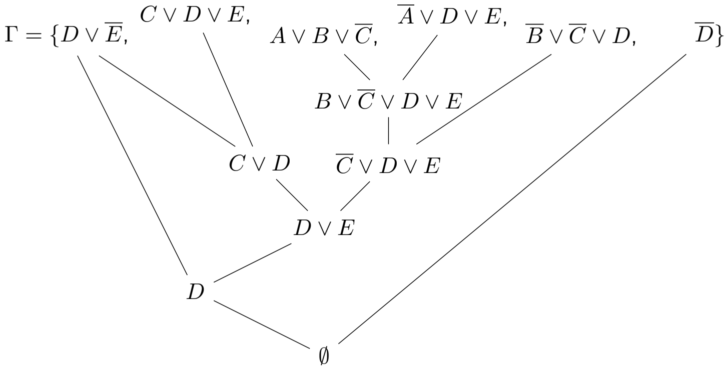

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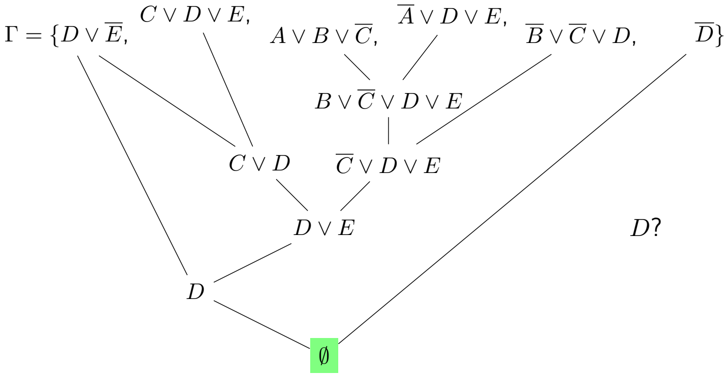

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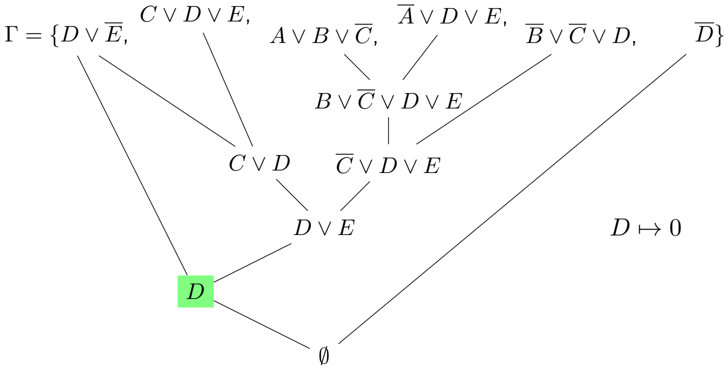

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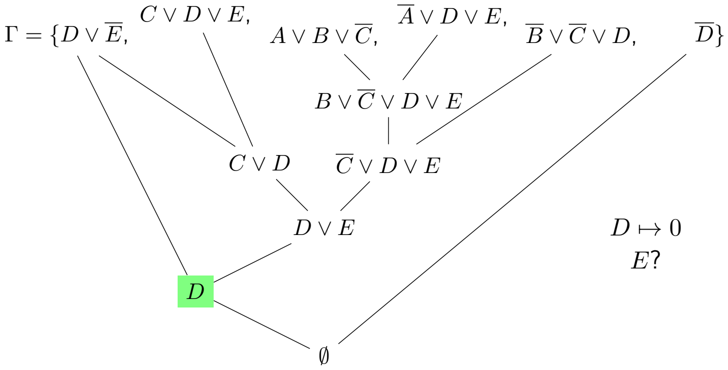

$D \mapsto 0$
E?

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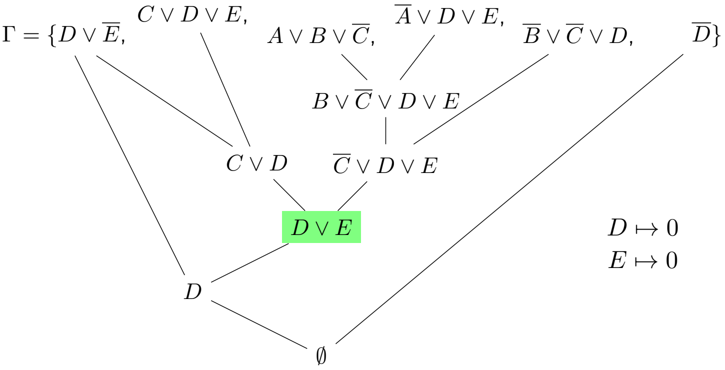

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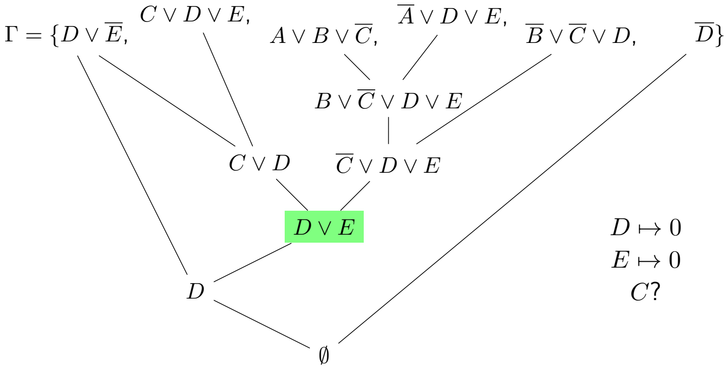

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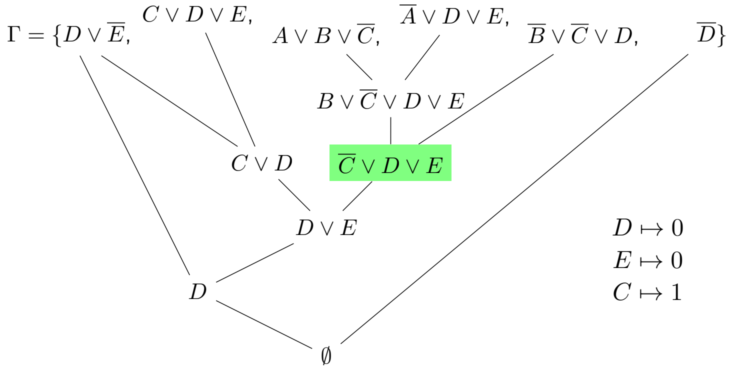

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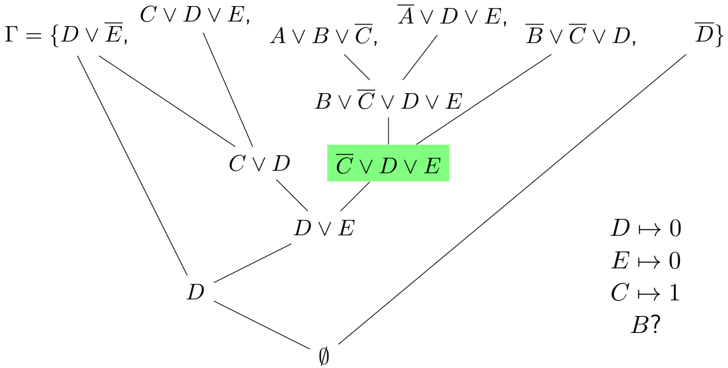

$D \mapsto 0$
$E \mapsto 0$
$C \mapsto 1$
B?

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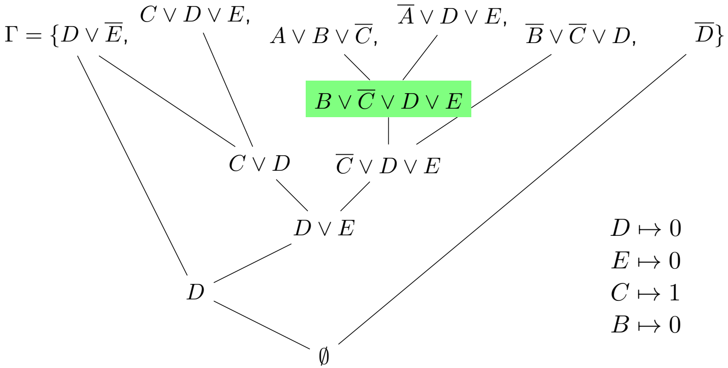

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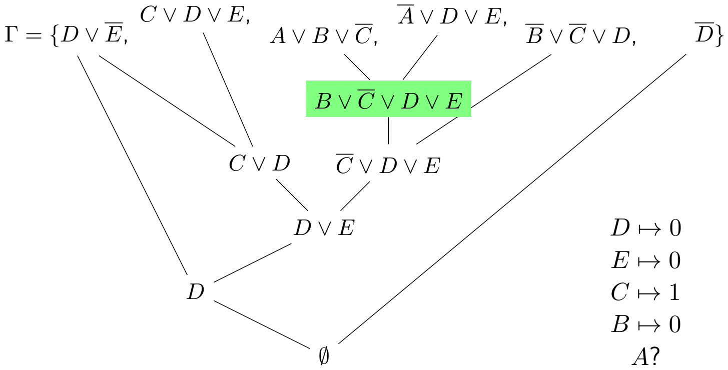

$D \mapsto 0$
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$C \mapsto 1$
$B \mapsto 0$
$A?$

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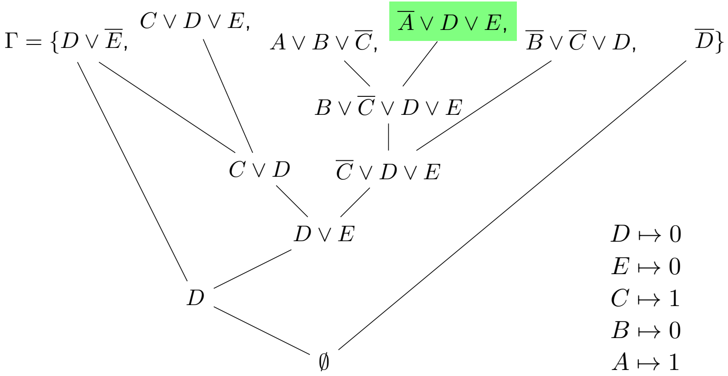

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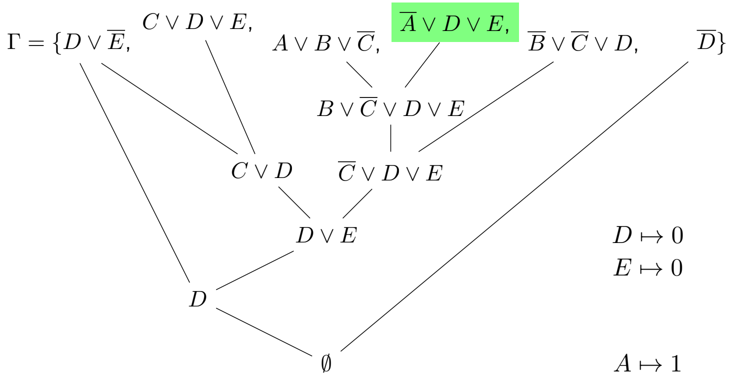
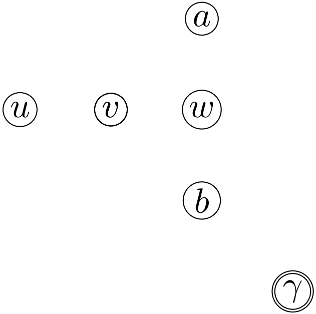


The k-pebble KAI-game



- nodes X, goal $\gamma \in X$

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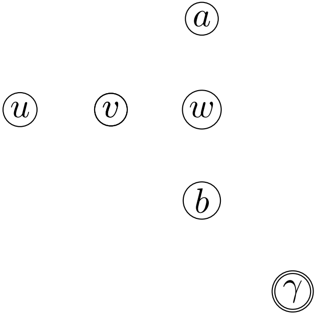

- nodes X, goal $\gamma \in X$
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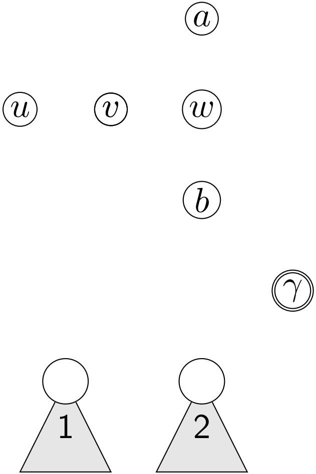
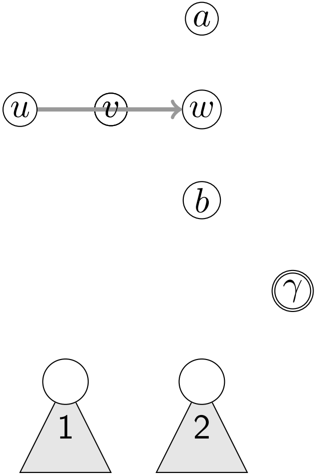


- nodes X, goal $\gamma \in X$
- k pebbles
- 2 players

The k-pebble KAI-game



- nodes X, goal $\gamma \in X$
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- rules (u, v, w)

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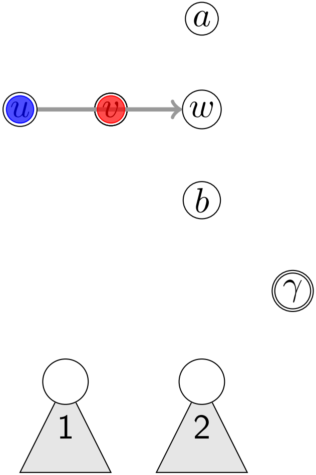

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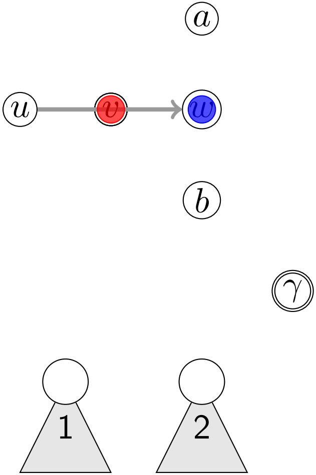

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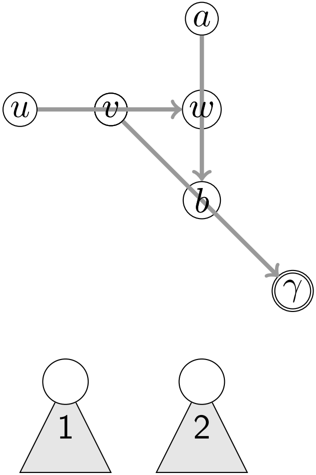

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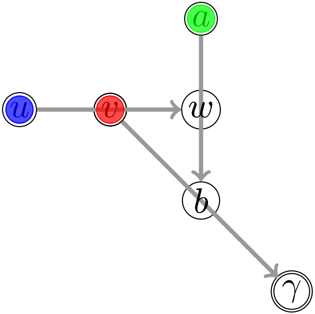

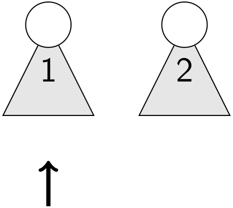

- nodes X, goal $\gamma \in X$
- k pebbles ● ● ●
- 2 players
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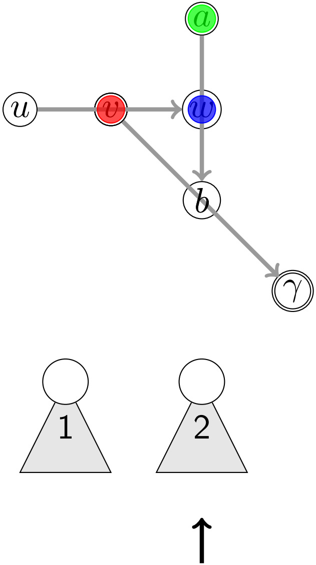

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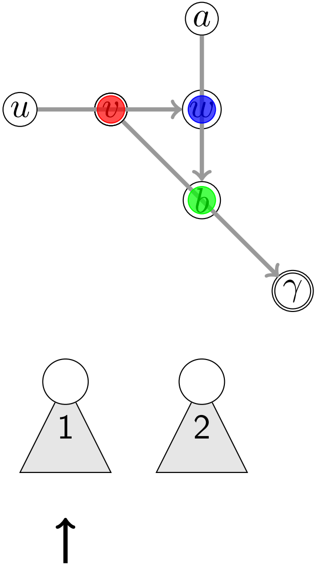

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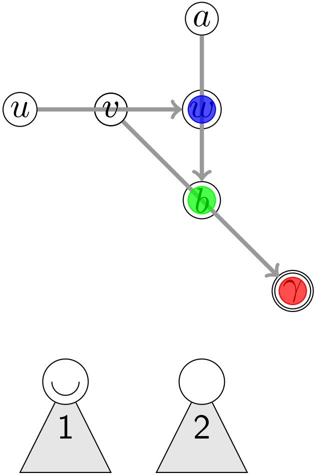

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KAI-game problem

Given an instance of the k-pebble KAI-game, does Player 1 win the game?

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Theorem (Kasai, Adachi, Iwata '79; '84)

The KAI-game problem is

- k *part of input:* EXPTIME-*complete.*
- k *fixed:* $\notin$ DTIME$(n^{\frac{k-2}{4}})$.

The reduction

We reduce the k-pebble KAI-game on I to the Boolean existential $(k+2)$-pebble game on $\Gamma(I)$.

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- Player 1 wins on $I \iff$ Spoiler wins on $\Gamma(I)$

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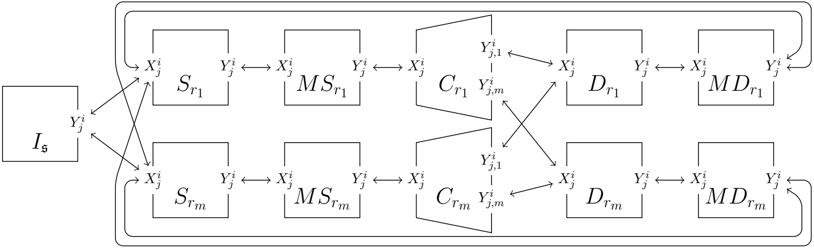

Thank you for your attention.