Distributional proving problems, and beyond

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(based on results of
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Distributional proving problems

Distributional proving problem (D, L) consists of

- a language L of "theorems",
- a polynomial-time samplable distribution $D = \{D_n\}_{n \in \mathbb{N}}$ on $\overline{L}$.

Motivation:

- a small (wrt D) amount of wrong theorems is acceptable;
- not interested in what happens on statements that are not claimed;
- polynomial-time samplable distributions are concentrated on **NP** languages, thus the definition is natural for $L \in$ **co-NP**.

Heuristic proof systems

- Allow probabilistic proof verification (with bounded error).
- Allow small number of false theorems (unbounded error there) according to D.
- Heuristic computation: gets d on input and makes at most $\frac{1}{d}$ errors.

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Heuristic proof system for (D, L) is a polynomial-time Π such that

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(correctness) Most non-theorems don't have such proofs:

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(completeness) There is a proof accepted whp:

$$\forall x \in L \; \forall d \in \mathbb{N} \; \exists w \quad \Pr\{\Pi(x, w, d) = 1\} > \tfrac{1}{2}.$$

(Such w is a Π-proof with confidence d.)

(correctness) Most non-theorems don't have such proofs:

$$\Pr_{r \leftarrow D_n}\{\exists w \; \{\Pr\{\Pi(r, w, d) = 1\} > \tfrac{1}{8}\}\} < \tfrac{1}{d}.$$

Turning AM protocols into heuristic proof systems

- Assume $L \in \mathbf{AM}$.
  (E.g., $L =$GNI, D samples random isomorphic graphs.)
- Consider a protocol (A, M) for L
  (w.l.o.g., with perfect completeness and exponentially small error):

$$x \in L \implies \forall r\ \exists w\ A(x, w, r) = 1,$$
$$x \notin L \implies \Pr_r\{\exists w\ A(x, w, r) = 1\} < 2^{-|x|}.$$

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- Consider $L' = \{(x, r) \mid x \in L\}$, where the length of r is enough to make the public random choices.
- Consider $D' = D \times U$, where D is any "original" distribution on $\overline{L}$ and U is the uniform distribution.

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Theorem (Itsykson, Sokolov)

(1) (L', D') has a polynomially bounded heuristic p.s.
(2) if $L' \in \mathbf{NP}$, then $L \in \mathbf{NP}$.

- Proof: Simulate A using r.

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Question: AM protocols make deterministic (heuristic) proof systems with very small error. Suggest another example: make use of randomness and larger error (dependent on the running time), go outside **AM**.

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Question: Is there a heuristic algorithm for (L', D')?
Answer: ~~We don't know.~~
Implies randomized heuristic algorithm for (L, D).

Heuristic acceptors

Definition

(Classical) acceptor A for L:

(completeness) A accepts every $x \in L$.

(correctness) A does not stop on any $x \notin L$.

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- Time $\tau_A(x, d)$ is a random variable.
- For random variable X, define $\mu^{(p)}[X] = \min\{T : \Pr[X \geq T] \geq p\}$.
- $t_A(x) = \mu^{(1/2)}[\tau_A(x, d)]$ is the median running time of $A(x, d)$.

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Theorem

*$\exists$ polynomial-time samplable D $\exists L \in$ **co-NP** $\nexists$ polynomial-time heuristic acceptor for (D, L) $\iff$ $\exists$ infinitely-often one-way function.*

Optimal heuristic acceptor

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(Classical) acceptor S simulates W if there is a polynomial p such that $\forall x \in L, \quad t_S(x) \leq p(t_W(x) \cdot |x|))$.

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Heuristic acceptor S simulates W if there are polynomials p and q such that $\forall x \in L, \forall d \in \mathbb{N}, \qquad t_S(x,d) \leq \max_{d' \leq q(d \cdot |x|)} p(t_W(x,d') \cdot |x| \cdot d)$.

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- For each $i \leq \log |x|$ in parallel:
  1. Execute $A_i(x,d')$.

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Here $d' = 4d|x|$, $k = 2d^3|x|^3$, $\delta = \frac{1}{2d|x|}$.

Optimal proof systems

- A proof system Σ simulates a proof system Ω iff Σ-proofs are at most as long as Ω-proofs (up to a polynomial p):

$$\forall F \in L \quad |\text{shortest } \Sigma\text{-proof of } F| \leq p(|\text{shortest } \Omega\text{-proof of } F|, |F|).$$

- p-simulation is a constructive version: For any w-size Ω-proof, one can compute a $p(w)$-size Σ-proof in polynomial time.
- (p-)optimal proof system (p-)simulates any other proof system.
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Theorem

$\exists$ ***p-optimal proof system*** $\iff$ $\exists$ ***optimal acceptor.***

For **TAUT**: [Krajícek, Pudlák].
For paddable languages: [Messner].
For **co-NP**-complete languages: [Chen, Flüm, Müller].

Simulations

- pointwise simulation $\mathcal{A} \prec \mathcal{B}$:
  $\exists$ polynomial p $\forall x$

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  $\forall \epsilon > 0$ $\exists c > 0$

$$\mathop{\mathbf{E}}_{x \leftarrow D_n}[{t_{\mathcal{A}}}^c(x)] = O(n \mathop{\mathbf{E}}_{y \leftarrow D_n}[{t_{\mathcal{B}}}^\epsilon(y)])$$

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- (weaker) simulation scheme:
  simulate everywhere except for the set of D-prob. $1/2d$.
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  - worst-case optimal acceptor for **NP**-complete problems:
    Levin's universal search + self-to-decision reduction:
    On input x, run $|x|$ algorithms in parallel:
    1. $A_1(x)$ (brute-force search); output the result;
    2. $A_2(x)$; check the solution; output if it's correct;

    ... ...

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    Not a pointwise optimal acceptor for **co-NP** problems;

    Not a pointwise optimal acceptor for **NP** problems;

    Main obstacle: how to verify a 1-bit answer to a decision problem?

    Worst-case optimal acceptor for **NP**-complete problems:
    extract satisfying assignment for F by queries to $F[v = 0]$, $F[v = 1]$, ...

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  - worst-case (and stronger) optimal randomized acceptor for GNI:
    verification by Goldwasser-Micali-Sipser protocol.

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  - pointwise-optimal acceptor for Time(f)-immune sets [Messner],
    pointwise-optimal algorithm for bi-immune sets [Chen,Flum,Müller].

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- Distributional problem (D, L): is $x \in L$ with accuracy d?
  Complexity measure = time(n, d).
  Errorless average-case complexity: count **E** or give up with D-prob. $1/d$.
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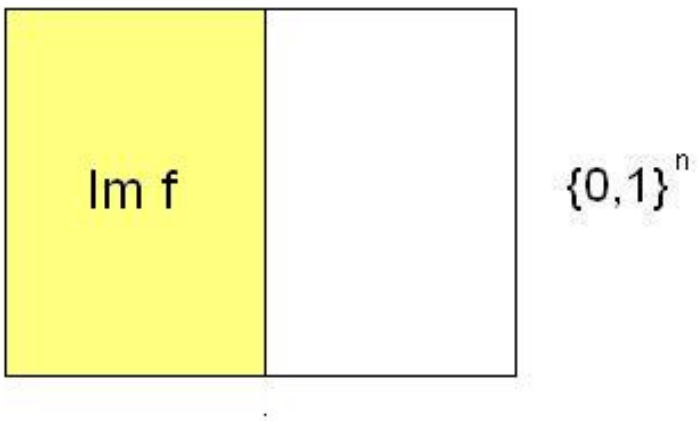
  - "scheme-optimal" deterministic *algorithm* for —"—"—.
- Distributional proving problem (D, L): $\mathbf{supp}\, D \subseteq \overline{L}$.
  Solved by heuristic acceptors, may allow false positives only.
  - pointwise optimal randomized heuristic acceptor for p.-t.s. D, r.e. L

Derandomization

Deterministic *scheme*-optimal acceptor for $(U, \overline{\mathbf{Im}\, f})$, where...



- $f : \{0,1\}^* \to \{0,1\}^*$,
- $|f(x)| = |x| + 1$,
- f is injective,
- f is polynomial-time computable.

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- Use pseudorandom graph based on expanders.
- The input is a source of randomness!
- Not optimal when the simulated algorithm is erroneously disqualified.

Definition

Simulation **scheme** of A by A':

Simulate everywhere except for the fraction $\frac{1}{2d}$:

$\exists$ polynomials p, q $\forall n, d \in \mathbb{N}$

$$\Pr_{x \leftarrow D_n}[t_A(x, d) \leq p(n \cdot d \cdot t_{A'}(x, q(n, d)))] \geq 1 - \frac{1}{2d},$$

$$q(n, d) \geq 2d.$$

Graph nonisomorphism

$\mathsf{GNI} = \{(G_1, G_2) \mid G_1 \not\cong G_2, |V(G_1)| = |V(G_2)|\}$,

- n is the number of vertices,
- G^π is the result of permuting $V(G)$ by $\pi \in S_n$.

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Recall **two-round interactive protocol** for GNI [Goldreich, Micali, Wigderson, 1987]:

- Prover claims that $G_1 \not\cong G_2$;
- Verifier picks random $i \in \{1, 2\}$, $\pi \in S_n$ and sends G_i^π;
- Prover sends j;
- Verifier accepts if $i = j$.

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If the claim is wrong, Verifier rejects with probability $\geq 1/2$.

Correcting a GNI algorithm

$\mathsf{SelfCorrect}_{A,N}$, corrects any (randomized) algorithm A:

- Run $N+1$ instances of A in parallel for random $\pi_{ij} \in S_n$:
  - $A(G_1{}^{\pi_{11}}, G_1{}^{\pi_{12}})$
  - $A(G_1{}^{\pi_{21}}, G_1{}^{\pi_{22}})$
  - $\ldots$
  - $A(G_1{}^{\pi_{N1}}, G_1{}^{\pi_{N2}})$
  - $A(G_1{}^{\pi_{N+1,1}}, G_2{}^{\pi_{N+1,2}})$
- Return 1 if the last instance was the fastest; otherwise diverge.

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Lemma

- *If* $G_1 \simeq G_2$, *then* $\Pr[\mathit{accept}] \leq \frac{1}{N+1}$.
- *If* $G_1 \not\simeq G_2$ *and* A *errs with probability* $\leq \frac{1}{2^n}$, *then* $\Pr[\mathit{accept}] \geq 1 - \frac{N+1}{2^n}$.

Optimal acceptor for GNI

Algorithm $Opt(G_1, G_2)$:

- Execute in parallel:
  - $A_1(G_1, G_2)$ (brute-force search),
  - 3 times SelfCorrect$_{A_2,30n}(G_1, G_2)$,
  - 3 times SelfCorrect$_{A_3,30n}(G_1, G_2)$,
  - ...
  - 3 times SelfCorrect$_{A_n,30n}(G_1, G_2)$.
- Accept if any of the $3n + 1$ parallel threads accepts.

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For any randomized acceptor A *for GNI* $\exists$ *polynomial* p *such that* $\forall x \in GNI,\ t_{Opt}(x) \leq p\left(\mu^{(1/4)}_{y \leftarrow U(C_x)}[\tau_A(y)]\right)$, *where* $C_{(G_1,G_2)} = \{(G_1^{\pi_1}, G_2^{\pi_2}) \mid \pi_1, \pi_2 \in S_n\}$ *is a cluster of* (G_1, G_2).

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Corollary

Opt is average-case optimal provided D is uniform on every cluster.

From acceptors to proof systems

Definition

L is paddable if there is an injective non-length-decreasing polynomial-time padding function $\mathrm{pad}_L : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ that is polynomial-time invertible on its image and such that $\forall x, w\ (x \in L \iff \mathrm{pad}_L(x, w) \in L)$.

Optimal proof [Messner, 99]:

- A proof π of x in some system Π;
- padding.

Verification:

- run optimal acceptor on $\mathrm{pad}_L(x, \pi)$;
- for a correct proof π, it accepts in a polynomial time because for a correct system Π, the set $\{\mathrm{pad}_L(x, \pi) \mid x \in L,\ \Pi(x, \pi) = 1\} \subseteq L$ can be accepted in a polynomial time.

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Applicability:

- Messner's proof goes for randomized algorithms.
- Does not go for heuristic, average-case algorithms.

Open questions

- $\exists$ optimal proof system $\iff$ $\exists$ optimal heuristic acceptor;
- $\exists$ optimal heuristic proof system $\overset{?}{\iff}$ $\exists$ optimal heuristic acceptor;
- $\exists$ optimal proof system with advice $\overset{?}{\iff}$ $\exists$ optimal acceptor with advice;
- $\exists$ average-case optimal acceptor?
- $\exists$ optimal acceptor for GNI or any other $\mathbf{co\text{-}NP} \setminus \mathbf{P}$ problem?
- $\exists$ optimal proof system for any problem outside $\mathbf{P}$?
- $\exists (D, L) \in (\mathbf{co\text{-}NP}, \text{PSamplable})$ with no polynomially-bounded heuristic proof system $\iff$?
- **AM** protocols make deterministic (heuristic) proof systems with very small error; suggest another example: randomized and with larger error.

CSR-2013

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