On the Virtue of Succinct Proofs: Amplifying Communication Complexity Hardness to Time-Space Trade-offs in Proof Complexity

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Limits of Theorem Proving
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Joint work with Trinh Huynh

The SAT Problem in Theory and Practice

- SAT NP-complete and so probably intractable in worst case
- But enormous progress on applied algorithms last 10-15 years
- Surprising fact 1: State-of-the-art SAT solvers can deal with real-world instances containing millions of variables
- Surprising fact 2: Best SAT solvers today still based on methods from early 1960s (i.e., DPLL and resolution)
- Algebraic and geometric methods more efficient in theory but **not** so far in practice

SAT Solving and Proof Complexity

SAT solving

- Constructive (almost deterministic) algorithms
- Key resources for solvers: time and memory
- Ideally minimize simultaneously

Proof complexity

- Study proofs, i.e., nondeterministic algorithms
- Complexity measures: proof size and proof space
- Lower bounds for optimal algorithms

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Hope to understand potential and limitation of SAT solvers by studying corresponding proof systems

Complexity measures also natural and interesting in their own right

This talk: Size-space trade-offs for algebraic and geometric systems

Outline

1. Proof Complexity
   - Preliminaries
   - Previous Work
   - Our Results

2. Tools and Techniques
   - Pebbling
   - Communication Complexity
   - Lifting
   - Critical Block Sensitivity

3. Open Problems

Some Terminology and Notation

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: all clauses of size $\leq k = \mathcal{O}(1)$
- Goal: Refute given CNF formula (i.e., prove it is unsatisfiable)
- Refer to clauses of CNF formula as axioms
  (as opposed to conclusions derived from these clauses)
- All formulas in this talk are k-CNFs
  (cleanest and most interesting case)

The Theoretical Model

- Proof system operates with lines of some syntactic form
- Proof/refutation is "presented on blackboard"
- Derivation steps:
  - Write down axiom clauses of CNF formula being refuted (as encoded by proof system)
  - Infer new lines by deductive rules of proof system
  - Erase lines not currently needed (to save space on blackboard)
- Refutation ends when contradiction is derived

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Complexity Measures: Length, Size and Space

Length

\# derivation steps

Size

$\approx$ total # symbols in proof counted with repetitions

Space

$\approx$ max size of blackboard to carry out proof
(e.g., space 3 for this blackboard)

$$x \lor \overline{y} \lor z$$
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Note that:

1. These are somewhat informal definitions — see paper for (standard) details
2. Length and size can be very different — won't really distinguish between them too much in this talk

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Basis for the most successful SAT solvers to date
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- Optimal (exponential) lower bounds on size
  [Urquhart '87; Chvátal & Szemerédi '88]
- Optimal (linear) lower bounds on clause space
  [Torán '99; Alekhnovich, Ben-Sasson, Razborov & Wigderson '00]
- Strong size-space trade-offs
  [Ben-Sasson & N. '11; Beame, Beck & Impagliazzo '12]

Polynomial Calculus (or Actually PCR [ABRW '00])

Clauses interpreted as polynomial equations over finite field

E.g., $x \lor y \lor \overline{z}$ translated to $x'y'z = 0$

Show no common root by deriving $1 = 0$

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Boolean axioms $\dfrac{}{x^2 - x = 0}$ *Negation* $\dfrac{}{x + x' = 1}$

Linear combination $\dfrac{p = 0 \quad q = 0}{\alpha p + \beta q = 0}$ *Multiplication* $\dfrac{p = 0}{xp = 0}$

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- Optimal (exponential) lower bounds on size [Alekhnovich-Razborov '01] and others
- Only recently lower bounds on monomial space for k-CNFs [Filmus, Lauria, N., Ron-Zewi & Thapen '12] building on [ABRW '00]
  Very recent optimal bounds in [Bonacina & Galesi '13]
- No size-space trade-offs

Cutting Planes

Clauses interpreted as linear inequalities
E.g., $x \lor y \lor \overline{z}$ translated to $x + y + (1 - z) \geq 1$
Show inconsistent by deriving $0 \geq 1$

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Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

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- Only one (exponential) lower bounds on size [Pudlák '97]
- No lower bounds on line space
- No size-space trade-offs

Trade-offs for Polynomial Calculus and Cutting Planes

We make some progress on understanding space and size-space trade-offs in polynomial calculus and cutting planes

Theorem (Informal)

There are k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\Theta(n)$ such that

- *resolution can refute F_n in length $\mathcal{O}(n)$ (and hence so can polynomial calculus and cutting planes)*
- *any polynomial calculus or cutting planes refutation of F_n in length L and space s must have*

$$s \log L \gtrsim \sqrt[4]{n}$$

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$$s \log L \gtrsim \sqrt[4]{n}$$

Nice bonus: lower bounds hold for semantic versions of proof systems where anything implied by blackboard can be inferred in just one step

Proof Ingredients

- Pebbling
- Communication complexity
- Lifting
- Critical block sensitivity

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Questions about time-space trade-offs fundamental in theoretical computer science

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In particular, well-studied (and well-understood) for pebble games modelling calculations described by DAGs ([Cook & Sethi '76] and many others)

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

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- Time needed for calculation: # pebbling moves
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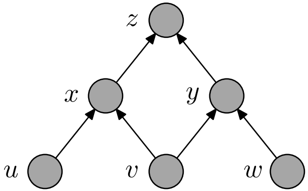
Some quick graph terminology

- DAGs consist of vertices with directed edges between them
- vertices with no incoming edges: sources
- vertices with no outgoing edges: sinks

The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	0
Current # pebbles	0
Max # pebbles so far	0

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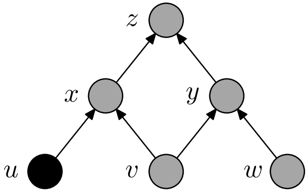

# moves	1
Current # pebbles	1
Max # pebbles so far	1

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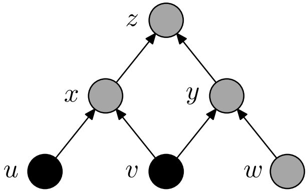

# moves	2
Current # pebbles	2
Max # pebbles so far	2

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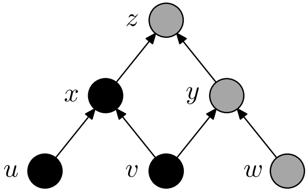

# moves	3
Current # pebbles	3
Max # pebbles so far	3

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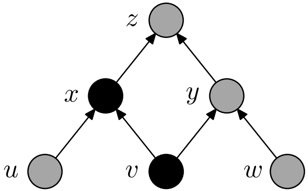

# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex

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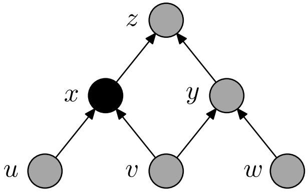

# moves	5
Current # pebbles	1
Max # pebbles so far	3

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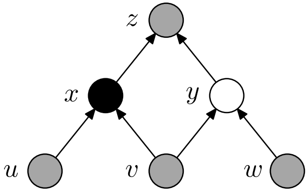

# moves	6
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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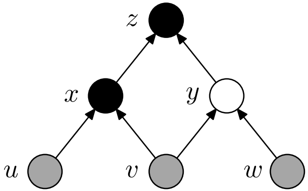

# moves	7
Current # pebbles	3
Max # pebbles so far	3

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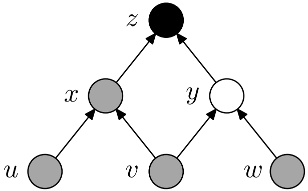

# moves	8
Current # pebbles	2
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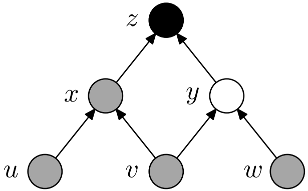

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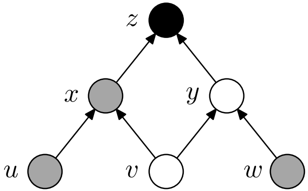

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Current # pebbles	3
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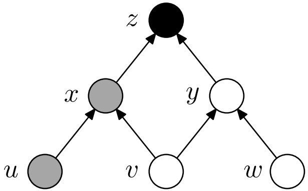

# moves	10
Current # pebbles	4
Max # pebbles so far	4

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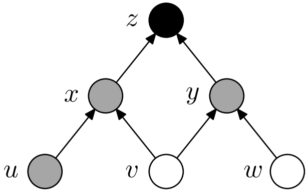

# moves	11
Current # pebbles	3
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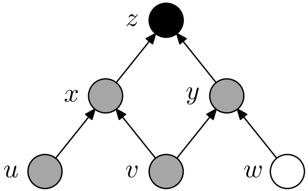

# moves	12
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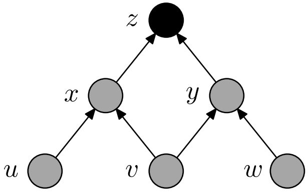

# moves	13
Current # pebbles	1
Max # pebbles so far	4

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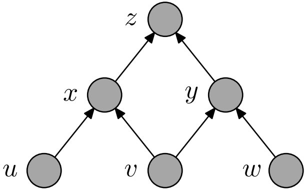
Pebbling Contradiction

CNF formulas encoding pebble game played on DAG G

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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- truth propagates upwards
- but sink is false

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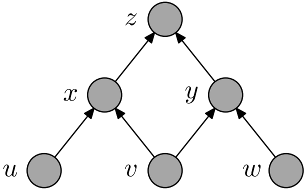

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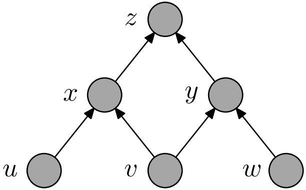

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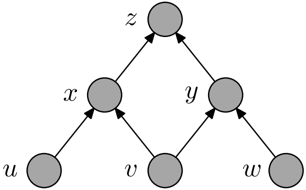

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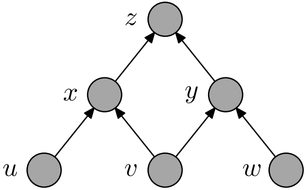
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



- sources are true
- truth propagates upwards
- but sink is false

Appeared in various contexts in [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and other papers

Used to study size and space in resolution in [N. '06, N. & Håstad '08, Ben-Sasson & N. '08, '11]

Two-Player Randomized Communication Complexity

- Alice has private input x and private source of randomness
- Bob has private input y and private source of randomness
- Both have unbounded computational powers
- Want to compute $f(x, y)$ by sending messages back and forth
- Output correct for any x and y except with error probability ε
- Communication cost: max # bits communicated on any x and y

Falsified Clause Search Problem

Fix:

- unsatisfiable CNF formula F
- (devious) partition of $Vars(F)$ between Alice and Bob

Falsified clause search problem $Search(F)$

Input: Assignment α to $Vars(F)$ split between Alice and Bob

Output: Clause $C \in F$ such that $\alpha(C) = 0$

Actually, computing not function but relation — more about that later

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Evaluate blackboard configurations of a refutation of F under α

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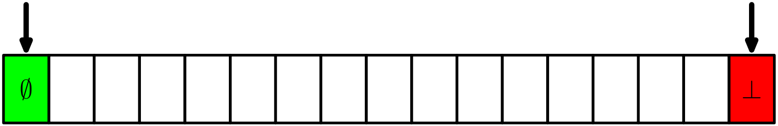

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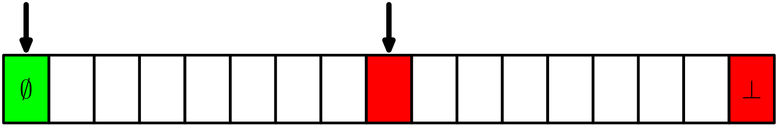

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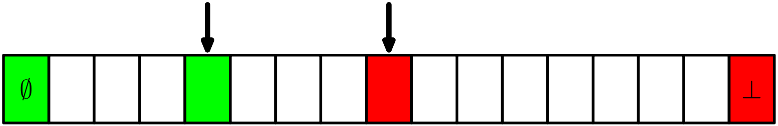

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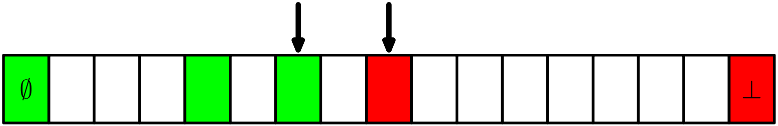

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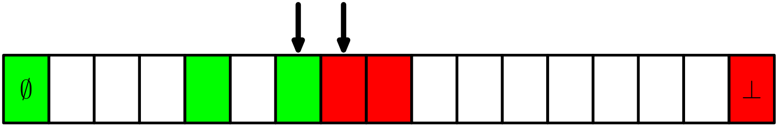

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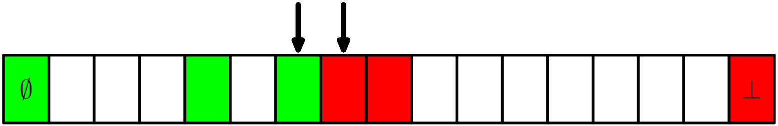

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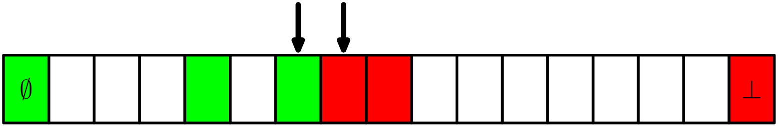

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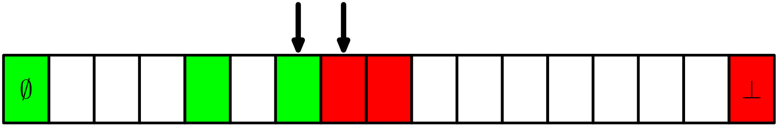

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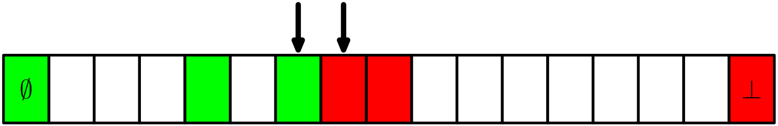

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(E.g. for polynomial calculus Alice and Bob simply evaluate their part of each monomial and exchange values — cutting planes bit more involved but can be done)

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Construct hard communication problems by "hardness amplification" using lifting

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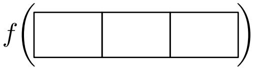

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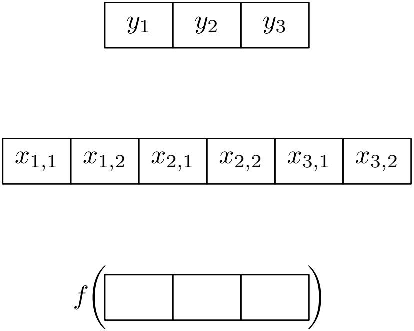

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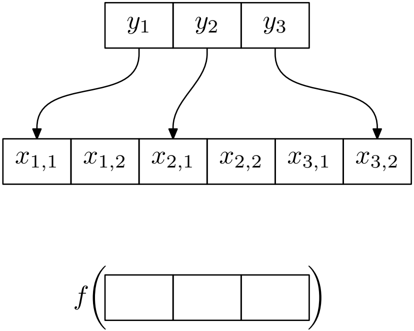

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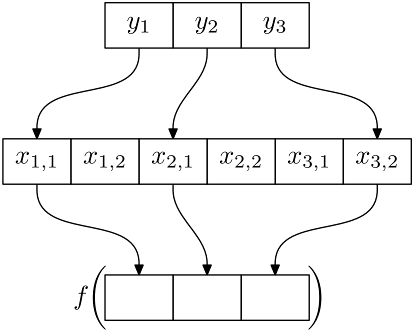

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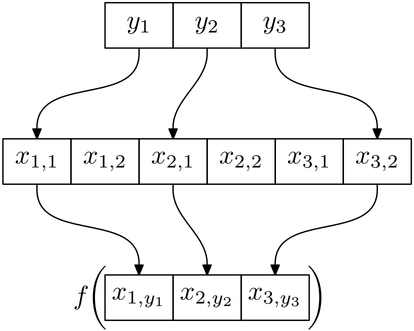
Length-ℓ lifting of f defined as
$Lift_\ell(f)(x,y) := f(x_{1,y_1}, \ldots, x_{m,y_m})$



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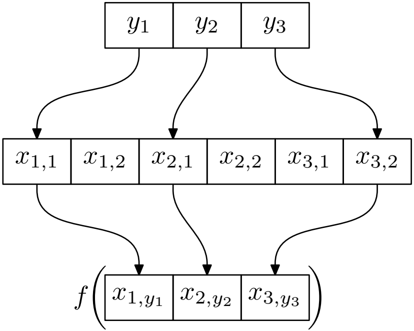

Idea borrowed from [Beame, Huynh & Pitassi '10]

Critical Block Sensitivity of Search Problems

- Block sensitivity of f on α: $\#$ disjoint blocks of α that flip f if flipped
- Problem: falsified clause search problem defines relation, not function
- Study block sensitivity of search problems
- In addition restrict to critical inputs (where relation is “function-like” in that there is only one right answer)
- Prove randomized communication complexity lower bounds in terms of critical block sensitivity of search problems
- Proof uses information-theoretic approach inspired by [Bar-Yossef, Jayram, Kumar & Sivakumar ’04]

Communication Complexity Results

We prove two technical lemmas:

Lemma 1

If critical block sensitivity of search problem S is large, then communication complexity of lifted search problem $Lift(S)$ is large

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Lemma 2

Search problems for pebbling formulas constructed from specific family of pyramid graphs have large critical block sensitivity

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- Encode lifting of search problem for CNF as new formula $\mathit{Lift}(F)$ (as in [Beame, Huynh & Pitassi '10])

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- But communication complexity of lifted search problem lower-bounded by critical block sensitivity (Lemma 1)
- Plug in lower bound for pyramid pebbling formulas (Lemma 2)
  $\Rightarrow$ trade-off for lifted pebbling formulas

Block Sensitivity in More Detail

Block sensitivity of f at α:

$$f\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \hline\end{array}\right) = 0$$

Block Sensitivity in More Detail

Block sensitivity of f at α:

$$f\left(\boxed{0 \mid 0 \mid 0 \mid 1 \mid 1 \mid 0 \mid 1 \mid 0 \mid 0 \mid 0 \mid 1 \mid 0}\right) = 1$$

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Block sensitivity of f at α:

$$f\left(\boxed{0\ |\ \mathbf{1\ |\ 1\ |\ 0}\ |\ 1\ |\ 0\ |\ 1\ |\ 0\ |\ 0\ |\ 0\ |\ 1\ |\ 0}\right) = 0$$

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$$f\left(\boxed{0}\,\boxed{1\;1\;0}\,\boxed{1\;0\;1}\,0\,\boxed{\color{red}1}\,0\;1\;0\right) = \color{red}{1}$$

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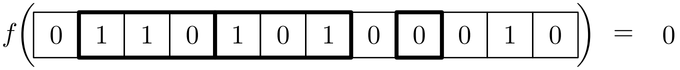

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f solves search problem $S \subseteq \{0,1\}^m \times Q$ if it holds that $(\alpha, f(\alpha)) \in S$

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$bs_{crit}(S)$: block sensitivity over critical assignments $\mathcal{A}$ of best f solving S

Lifting and Critical Block Sensitivity

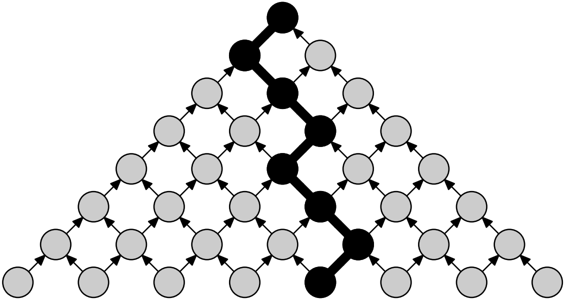
Lemma 2 (more formal version)

Suppose $S \subseteq \{0,1\}^m \times Q$ is a search problem and $\ell \geq 3$. Then any consistent randomized protocol solving $\mathit{Lift}_\ell(S)$, where Alice receives the selector y-variables and Bob receives the main x-variables, requires $\Omega(bs_{crit}(S))$ bits of communication.

Proof is by

- information theory tools
- direct sum theorem à la [BJKS04]

Critical Assignments for Pyramid Pebbling Formulas



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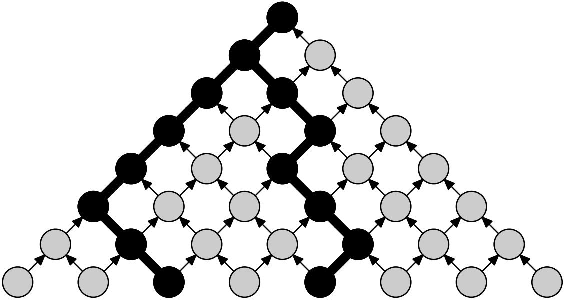

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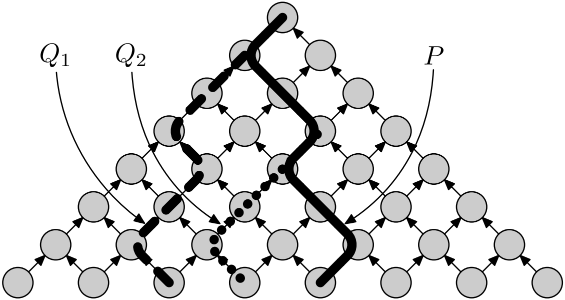
Bicritical assignments falsify two different paths
$\Rightarrow$ two possible correct answers

Path Graph

Build graph G such that

- vertices = source-to-sink paths P
- edge (P, Q) only if P and Q merge and stay together
- in addition, if (P, Q_1) and (P, Q_2) edges, then $Q_1 \cap Q_2 \subseteq P$
- G is undirected — (P, Q) edge only if (Q, P) edge



Dense Path Graph $\Rightarrow$ High Critical Block Sensitivity

Lemma 3

If $\exists$ path graph G with average degree d, then falsified clause search problem for pebbling formula has critical block sensitivity $\geq d/2$

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Lemma 4

For pyramid on n vertices, can get average degree $\Omega\left(\sqrt[4]{n}\right)$

More General Trade-offs?

Our proofs only work for formulas generated from pyramid graphs

For resolution, correspondence between pebbling and size-space trade-offs holds for arbitrary graphs

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Recently achieved for polynomial calculus in [Beck, N. & Tang '13]
Uses different techniques; in particular random restrictions
$\Rightarrow$ not tight results as for resolution, so room for further improvements

Still open for cutting planes (random restrictions don't work)

Unconditional Space Lower Bounds?

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Can log *length factor be removed from results to yield unconditional space lower bounds?*

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Again answer known to be "yes" for resolution

But [Beck, N. & Tang '12] still has log factor for polynomial calculus

Underlying question: For how wide a family of proof systems do pebbling properties of graphs carry over to CNF size-space trade-offs?

Take-Home Message

- Modern SAT solvers enormously successful in practice — key issue is to minimize time and memory consumption
- Modelled by proof size and space in proof complexity
- We show trade-offs indicating that simultaneous optimization impossible for well-known algebraic and geometric proof systems
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Thank you for your attention!