A Conditional Logic for Iterated Belief Revision

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Abstract. In this paper we propose a conditional logic IBC to represent iterated belief revision. We define an iterated belief revision system by strengthening the postulates proposed by Darwiche and Pearl [3]. First, following the line of Darwiche and Pearl, we modify AGM postulates to make belief revision a function of epistemic states rather than of belief sets. Then we propose a set of postulates for iterated revision which, together with the (modified) AGM postulates, entail Darwiche and Pearl's ones.

The conditional logic IBC has a standard semantics in terms of selection function models and provides a natural representation of epistemic states. IBC contains conditional axioms, corresponding to the postulates for iterated revision. We provide a representation result, which establishes a one to one correspondence between iterated belief revision systems and IBC-models. We prove that Gärdenfors' Triviality Result does not apply to IBC.

1 Introduction

In [8] we have introduced a conditional logic BC to represent belief revision. The logic BC has a standard semantics in terms of possible worlds structures with a selection function and has strong similarities with Stalnaker's logic C2.

According to Ramsey's proposal [13] in order to decide whether to accept a conditional proposition $A > B$ (whose meaning is: "if A were true then B would be true") we should add the antecedent A to our belief set, changing it as little as possible, and then consider whether the consequent B follows. In the context of the theory of epistemic change, which has been developed by Gärdenfors [5, 7] together with Alchourrón and Makinson [1], this acceptability criterion is expressed by the well known Ramsey Test:

$A > B \in K$ iff $B \in K * A$,

where K represents a belief set (that is, a deductively closed set of sentences) and * represents a *Belief Revision operator*. The operator * transforms ("revises") a belief set K by adding a formula A in such a way that the resulting belief set, denoted by $K * A$, is consistent if so is A; moreover, $K * A$ is obtained by minimally changing K. Gärdenfors Alchourrón and Makinson have proposed a set of postulates, called AGM, which are intended to characterize any belief revision operator.

In spite of the similarities between the semantics of belief revision and the evaluation of conditionals, the above very intuitive acceptance principle leads to the well known Triviality Result by Gärdenfors, [5], that claims that no significant belief revision systems are compatible with the Ramsey Test.

In [8] we have devised a correspondence between belief revision systems and conditional logic that does not entail the Triviality Result. We have established a mapping between revision systems and the conditional logic BC by a representation result showing how each belief revision system determines a $BC-$structure, and how each $BC-$structure defines a belief revision system.

The aim of this paper is to explore if a similar correspondence with conditional logics can be obtained for *iterated* belief revision systems. Iterated belief revision has been widely investigated in recent years [2, 3, 11, 14]. In particular, it has been shown that the AGM postulates are too weak to ensure the rational preservation of conditional beliefs during the revision process. For this reason new postulates have been proposed which "characterize belief revision as a process which may depend on elements of an epistemic state that are not necessarily captured by a belief set" [3].

In this paper we introduce a conditional logic IBC to represent iterated belief revision. The logic IBC is an extension of the logic BC and it provides a natural representation of epistemic states. In IBC epistemic states are not introduced as new semantic objects, as it is done, for instance, by Friedman and Halpern in [4]. On the contrary, IBC-models are defined as standard possible worlds models with selection function, and each world carries with itself, so to say, all the information concerning an epistemic state: a belief set and a set of revision strategies. As pointed out in [3], "any such strategy encodes, and is equivalent to, a set of "conditional" beliefs, that is, beliefs that one is prepared to adopt conditioned on any hypothetical evidence". We identify an epistemic state with a set of *equivalent* worlds in a IBC-structure, and the revision strategies relative to that state simply with the conditional formulas holding in those worlds.

AGM postulates fail to properly regulate iterated revision, since they are one-step postulates: they only deal with the transformation of belief sets and do not deal with the transformation of revision strategies as encoded in epistemic states. On the opposite, a theory of iterated belief revision must not only account for the change of beliefs in face of new observations, but also for the change of the revision strategies, that is, of conditional beliefs.

In order to deal with iterated revision, the preservation of conditional beliefs has to be ruled as well. Boutilier [2] has proposed a belief revision operator, called *natural revision*, which guarantees that conditional beliefs are preserved as much as the AGM postulates permit. In [3] Darwiche and Pearl show that this solution may lead to counterintuitive results, since it might compromise the preservation of propositional beliefs. They adopt a more cautious approach which aims to preserve all those conditional beliefs that might not compromise the preservation of propositional beliefs. First, they propose a modification of the AGM postulates in which revisions are applied to epistemic states rather than to belief sets. Then they introduce four postulates regulating iterated revision and provide a representation theorem for them. However, they do not define a conditional logic

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for iterated revision, which is the main aim of this paper.

In the next section we introduce a set of postulates for iterated belief revision strenghtening the postulates proposed by Darwiche and Pearl [3]. We show that Darwiche and Pearl's postulates can be derived from ours. Moreover, we provide a concrete operator satisfying our postulates, thus proving their consistency. The proposed postulates are well suited to be mapped onto conditional axioms. In section 3 we define the conditional logic IBC which has strong similarities with Stalnaker's logic C2 and contains conditional axioms corresponding to the postulates for iterated revision. We also develop a semantic interpretation for the logic IBC in terms of possible worlds structures with a selection function. In section 4 we prove a representation theorem, establishing a mapping between iterated revision and conditional models: to each iterated belief revision system corresponds an $IBC-$structure and to each $IBC-$structure (satisfying the covering condition) corresponds an iterated belief revision system. Finally we prove that the logic IBC is non-trivial in the sense of [9].

2 Iterated Belief Revision

Alchourrón, Gärdenfors and Makinson in [1] have proposed a set of rationality postulates that any belief change operator must satisfy. In [1, 7, 6] they introduce the operations of *expansion* and *revision* on *belief sets* (that is, deductively closed set of propositional formulas). *Expansion* is the simple addition of a formula A to a belief set K, and it is defined by: $K + A = Cn(K \cup \{A\})$. *Revision* is the consistent addition of a formula A to a belief set K, denoted by $K * A$.

As pointed out in the introduction, several authors [3, 4] have recognized that AGM postulates are too weak to account for iterated revision: they only rule the preservation of propositional formulas while they do not say anything about the preservation of conditional beliefs.

In order to deal with iterated revision, we need a notion of *epistemic state*. An epistemic state has an associated belief set, but also an associated set of revision strategies that the agent wishes to employ in that state to accommodate new evidences. Such revision strategies can be regarded as conditional beliefs, and they can be different in two different epistemic states even when they share the same belief set.

An *iterated belief revision system* is a triple $\langle S, *, [\ \ ]\rangle$ where S is a non-empty set whose elements are called epistemic states, $* : S \times \mathcal{L} \rightarrow S$ is the revision operator defined by the postulates below, $[\ ] : S \rightarrow P(\mathcal{L})$ is a function that maps each epistemic state to a belief set (i.e. a deductively closed set of propositional formulas). Throughout this paper we shall consider *consistent* iterated belief revision systems, i.e. iterated belief revision systems that contain only consistent epistemic states, where an epistemic state is consistent if its associated belief set is consistent. Moreover we will only consider revisions with consistent formulas.

Let us now state our modified AGM postulates. Given $\langle S, *, [\ ]\rangle$, and $\Psi, \Phi \in S$, we have:

$(R*1)$ $A \in [\Psi * A]$
$(R*2)$ If $\neg A \not\in [\Psi]$, then $[\Psi * A] = [\Psi] + A$;
$(R*3)$ If A is satisfiable, then $[\Psi * A]$ is also satisfiable;
$(R*4)$ If $A_1 \equiv A_2$, then $\Psi * A_1 = \Psi * A_2$;
$(R*5)$ $[\Psi * (A \wedge B)] \subseteq [\Psi * A] + B$
$(R*6)$ If $\neg B \not\in [\Psi * A]$, then $[\Psi * A] + B \subseteq [\Psi * (A \wedge B)]$

We can observe that most of the revision postulates above are only concerned with the belief sets resulting from certain revisions: they impose that two different revisions of an epistemic state lead to the same belief set or to related belief sets. However, postulate (R*4), as well as postulate (A4) below, requires something more: given an epistemic state, its revision by the formula A_1 and by an equivalent formula A_2 leads to the same epistemic state. Therefore, our postulate $(R*4)$ is stronger than the original AGM postulate, but also than the corresponding Darwiche and Pearl's postulate, which says that the revision of an epistemic state by equivalent formulas gives the same belief set, but not necessarily the same epistemic state.

In addition to the postulates $(R*1)$,..., $(R*6)$ we introduce the following *postulates for iterated revision*. We assume that * associates on the left:

$(A1)$ If $B \models \neg A$, then $[\Psi * A * B] = [\Psi * B]$;
$(A2)$ If $A \in [\Psi * B]$, then $[\Psi * A * B] = [\Psi * B]$;
$(A3)$ If $\neg A \not\in [\Psi * B]$, then $[\Psi * A * B] \subseteq [\Psi * B] + A$;
$(A4)$ $\Psi * \top = \Psi$

Postulate $(A1)$ says that if two contradictory pieces of information, A and B, are successively learned, then the belief set obtained by the revision of Ψ by A and then by B does not depend on the false intermediate revision A. In particular, $(A1)$ implies that $[\Psi * A * \neg A] = [\Psi * \neg A]$. Postulate $(A2)$ says that if A is believed after the revision of the epistemic state Ψ by B, then the belief set obtained by the revision of Ψ by A and then by B is the same as the one obtained by the revision of Ψ by B. Postulate $(A3)$ says that if A is consistent with the belief set resulting from the revision of Ψ by B, then by revising Ψ by A and then by B we cannot conclude more than by adding A to the result of the revision of Ψ by B. According to postulate $(A4)$, the revision of an epistemic state with $\top$ does not affect the epistemic state.

Note that while postulate $(A4)$ (similarly to $(R*4)$) says that the two epistemic states $\Psi * \top$ and Ψ are the same, postulates $(A1)$, $(A2)$ and $(A3)$ say something about the relation between the two belief sets $[\Psi * A * B]$ and $[\Psi * B]$. But this does not say anything about the conditional beliefs true in the two states $\Psi * A * B$ and $\Psi * B$. Indeed, for postulate $(A2)$, it might be the case that, although A holds in the state $\Psi * B$, we do not want to forget that the revision of Ψ by A has preceded the revision by B. For instance, we might want to give the pieces of information a different reliability according to their insertion time.

It can be shown that the postulates introduced above are consistent[3].

In particular it is possible to define a revision operator, which is based on Spohn's ordinal conditional functions [7] and which satisfies the postulates. For a definition of Spohn's revision operator, we refer to section 4 where it is used to prove the non-triviality of our conditional logic.

We prove that, given the modified AGM postulates above, the following postulates for iterated revision introduced by Darwiche and Pearl [3]:

$(C1)$ If $B \models A$, then $[\Psi * A * B] = [\Psi * B]$;
$(C2)$ If $B \models \neg A$, then $[\Psi * A * B] = [\Psi * B]$;
$(C3)$ If $A \in [\Psi * B]$, then $A \in [\Psi * A * B]$;
$(C4)$ If $\neg A \not\in [\Psi * B]$, then $\neg A \not\in [\Psi * A * B]$;

[3] However if we also considered inconsistent epistemic states postulate (A4) would conflict with postulate (R*3). To see this, consider an inconsistent epistemic state Ψ, according to (A4) $\Psi * \top = \Psi$, whereas according to (R*3) $\Psi * \top$ is consistent and hence different from Ψ. We think that in this case it is not obvious which one of the two postulates should be discarded. A possible way-out could be to weaken postulate $(A4)$, by adding to $(A4)$ a precondition requiring Ψ to be consistent.

can be derived from postulates $(A1)$, $(A2)$, $(A3)$. Notice first that $(A1)$ is the same as $(C2)$.

Lemma 1 *$(C1)$, $(C2)$, $(C3)$ and $(C4)$ can be derived from $(A1)$, $(A2)$,$(A3)$ together with $(R*1)-(R*6)$.*

We cannot prove the converse of Lemma 1, since $(A4)$ and $(R*4)$ cannot be derived from Darwiche and Pearl's postulates. As a matter of fact, none of Darwiche and Pearl's postulates enforces the equality between epistemic states obtained through different revisions. However, we believe that postulates $(A4)$ and $(R*4)$ define natural properties of revision functions. $(A4)$ says that a revision with a tautology cannot change the epistemic state and its revision strategy (Ψ and $\Psi * \top$ will determine the same belief sets under any sequence of revisions). $(R*4)$ says that the syntactical form of the revision formula is irrelevant in determining the resulting epistemic state. The weaker form of $(R*4)$ adopted by Darwiche and Pearl only requires that the syntactical form of the revision formula is irrelevant in determining the resulting belief set. We can prove the following:

Lemma 2 *$(A1)$, $(A2)$ and $(A3)$ can be derived from $(C1)$, $(C2)$, $(C3)$ and $(C4)$ together with $(R*1)-(R*6)$.*

Lehmann in [11] has proposed a set of rationality postulates for iterated revision. In his framework he represents the sequence of revisions applied to the initial belief set by a sequence of formulas σ and denotes by $[\sigma]$ the resulting belief set. The belief set $[\sigma \cdot A]$ represents the result of revising the belief set $[\sigma]$ by the formula A. In his framework the revision of $[\sigma]$ by the formula A depends not only on the belief set $[\sigma]$, but also on σ, that is the sequence of revisions that leads to $[\sigma]$. This sequence of revisions plays the role of the epistemic states in our context.

As Lehmann shows, $(C1)$, $(C3)$, and $(C4)$ can be derived from his postulates. However, only a weaker version of $(C2)$ holds in his framework (namely $[\sigma \cdot \neg A \cdot A] \subseteq [\sigma] + A$), and $(C2)$ (or $(A1)$) cannot be derived from Lehmann's postulates. On the other hand, Lehmann's postulates $(I4)$ and $(I6)$ cannot be derived from our postulates (nor can they be derived from Darwiche and Pearl's). Let us consider, for instance, his postulate $(I4)$: if $A \in [\sigma]$ then $[\sigma \cdot \tau] = [\sigma \cdot A \cdot \tau]$, where σ and τ are sequences of formulas. This postulate could be restated in our notation as: if $A \in [\Psi]$, then $\Psi * A = \Psi$. It means that when A is believed in Ψ, $\Psi * A$ and Ψ are the same epistemic state. This property cannot be derived from our postulates from which a weaker property follows, namely if $A \in \Psi$, then $[\Psi * A] = [\Psi]$, which is a consequence of $(R*2)$[4].

Among the consequences that Lehmann proves from his postulates, is the property that a mild revision (that is a revision with a formula consistent with the current belief set) essentially fades away at the first severe revision (that is a revision with a formula which is inconsistent with the current belief set). This property contrasts with the Principle of Minimal Change and is intuitively unwanted. Such a property cannot be proved from our postulates (nor from Darwiche and Pearl's ones).

3 The Conditional Logic IBC

In this section we introduce the conditional logic IBC. We will use it to represent iterated belief revision systems.

Definition 3 The language $\mathcal{L}_>$ of logic IBC is an extension of the language $\mathcal{L}$ of classical propositional logic obtained by adding the conditional operator $>$. Let us define the following modalities:

$$\Box A \equiv \neg A > \bot$$
$$\Diamond A \equiv \neg(A > \bot).$$

We define the language of modal formulas $\mathcal{L}_\Box$ as the smallest subset of $\mathcal{L}_>$ including $\mathcal{L}$ and closed under $\neg, \wedge, \Box, \Diamond$[5]. The logic IBC contains the following axioms and inference rules:

(G I) $(CLASS)$ All classical axioms and inference rules;

(ID) $A > A$;

$(RCEA)$ if $\vdash A \leftrightarrow B$, then $\vdash (A > C) \leftrightarrow (B > C)$;

(RCK) if $\vdash A \rightarrow B$, then $\vdash (C > A) \rightarrow (C > B)$;

(G II) (DT) $((A \wedge C) > B) \rightarrow (A > (C \rightarrow B))$, for $A, B, C \in \mathcal{L}$;

(CV) $\neg(A > \neg C) \wedge (A > B) \rightarrow ((A \wedge C) > B)$, for $A, B, C \in \mathcal{L}$;

(G III) (BEL) $(A > B) \rightarrow \top > (A > B)$;

$(REFL)$ $(\top > A) \rightarrow A$;

(EUC) $\neg(A > B) \rightarrow A > \neg(\top > B)$;

$(TRANS)$ $(A > B) \rightarrow A > (\top > B)$;

(G IV) (MOD) $\Box A \rightarrow B > A$, where $A \in \mathcal{L}_\Box$;

$(U4)$ $\Box A \rightarrow \Box\Box A$, where $A \in \mathcal{L}_\Box$;

$(U5)$ $\Diamond A \rightarrow \Box\Diamond A$, where $A \in \mathcal{L}_\Box$.

(G V) $(C1)$ $\Box\neg(A \wedge B) \wedge \Diamond A \rightarrow [(A > B > C) \leftrightarrow (B > C)]$, where $A \in \mathcal{L}_\Box$ and $C \in \mathcal{L}$;

$(C2)$ $B > A \rightarrow [(A > B > C) \leftrightarrow (B > C)]$, where $A \in \mathcal{L}_\Box$ and $C \in \mathcal{L}$;

$(C3)$ $[\neg(B > \neg A) \wedge (A > B > C)] \rightarrow (B > (A \rightarrow C))$, where $A \in \mathcal{L}_\Box$ and $C \in \mathcal{L}$.

We have gathered the axioms in different groups. Axioms of (G I) are those of the basic conditional logic CK+ID. Axioms (DT) and (CV) define essential properties of the conditional operator; they are part of the axiomatization of Stalnaker's logic (C2). We will come back to this point.

Axioms of (G III) are motivated by the introduction of the modal operator $\top > A$, whose meaning is "A is believed". The other axioms of this group (the last two for $A = \top$) give to this belief operator the properties of an S5 modality.

Similarly, axioms of (G IV) define a necessity operator $\Box$ and give it S5-properties. Axiom (MOD) governs the relation between $\Box$ and the conditional operator.

Axioms of (G V) encodes our postulates for iterated revision $(A1)$, $(A2)$,$(A3)$ by conditional axioms.

As mentioned above, (ID), (DT), (CV), (MOD) belong to the axiomatization of Stalnaker's logic $C2$ (see [12]). Stalnaker's logic contains also other axioms such as (CS) and (CEM); these axioms can be derived from the axiomatization above if we add axiom $A \rightarrow (\top > A)$ ("everything true is believed"), that we clearly do not want.

Moreover, it must be noticed that we have put restrictions on some axioms, by requiring that they only hold for formulas ranging over $\mathcal{L}$ rather than any conditional formula in $\mathcal{L}_>$. These restrictions are motivated by the fact that our logic is intended to model the revision postulates, and some of them, such as (A2) and (A3), only put requirements on the belief sets (but not on the epistemic states). This is also true for the semantic conditions (CV) and (DT), which are used to represent postulates (R*6) and (R*2).

[4] From our postulates we can prove $\Psi * A = \Psi$ only for $A = \top$.

[5] We assume that the conditional $>$ has higher precedence than the material implication $\rightarrow$.

We develop a semantics for the logic IBC in the style of standard Kripke-like semantics for conditional logics. Our structures are possible world structures equipped with a selection function [12]. Intuitively, the selection function, call it f, given a formula A and a world w, picks up the *most preferred* or closest worlds to w, denoted by $f(A, w)$, which satisfy A (if any). To evaluate a conditional $A > B$ in a world w we check if B holds in all worlds in $f(A, w)$. Different logics are obtained by imposing conditions on the selection function.

In our case, there is an intuitive correspondence between iterated belief systems and selection function models satisfying the properties of the next definition. The idea is that an epistemic state Φ can be represented by any set of *equivalent* worlds (in the sense defined below) which evaluate conditional formulas in the same way. Given a formula A, the selection function associates to two equivalent worlds w_1 and w_2 the same set of 'most-preferred' worlds $f(A, w_1) = f(A, w_2)$, and the worlds in this set are all equivalent among themselves. Thus $f(A, w_1)$ represents an epistemic state. We can see how the selection function can be used to specify a revision operator: the revision of an epistemic state Φ (a set of worlds) by A is simply the epistemic state $f(A, w)$, for any $w \in \Phi$ (it does not depend on the choice of w). In the next section we will see how each selection function model determines an iterated belief revision system and each iterated belief revision system determines a selection function model.

Definition 4 An IBC-structure M has the form $\langle W, f, [[]]\rangle$, where W is a non-empty set, whose elements are called possible worlds, f is a function of type $\mathcal{L}_> \times W \rightarrow 2^W$ and is called a selection function, $[[]] : \mathcal{L}_> \rightarrow P(W)$ is a valuation function satisfying the following conditions:

$(\bot)$ $[[\bot]] = \emptyset$
$(\wedge)$ $[[A \wedge B]] = [[A]] \cap [[B]]$
$(\neg)$ $[[\neg A]] = W - [[A]]$
$(>)$ $[[A > B]] = \{w : f(A, w) \subseteq [[B]]\}$.

The above definition is extended to the classical connectives $\vee, \rightarrow, \leftrightarrow$, by the usual classical equivalences. Let $Prop(S) = \{A \in \mathcal{L}: S \subseteq [[A]]\}$. We assume that the selection function f satisfies the following properties:

$(S-ID)$ $f(A, w) \subseteq [[A]]$;
$(S-RCEA)$ if $[[A]] = [[B]]$ then $f(A, w) = f(B, w)$
$(S-DT)$ $Prop(f(A \wedge C, w)) \subseteq Prop(f(A, w) \cap [[C]])$, for $A, C \in \mathcal{L}$;
$(S-CV)$ $f(A, w) \cap [[C]] \neq \emptyset \rightarrow Prop(f(A, w) \cap [[C]]) \subseteq Prop(f(A \wedge C, w))$, for $A, C \in \mathcal{L}$;
$(S-REFL)$ $w \in f(\top, w)$;
$(S-TRANS)$ $x \in f(A, w) \wedge y \in f(\top, x) \rightarrow y \in f(A, w)$;
$(S-EUC)$ $x, y \in f(A, w) \rightarrow x \in f(\top, y)$
$(S-BEL)$ $w \in f(\top, y) \rightarrow f(A, w) = f(A, y)$
$(S-MOD)$ If $f(B, w) \cap [[A]] \neq \emptyset$, then $f(A, w) \neq \emptyset$, where $A \in \mathcal{L}\Box$.
$(S-UNIV)$ if $[[A]] \neq \emptyset$, $\exists B$ such that $f(B, w) \cap [[A]] \neq \emptyset$, where $A \in \mathcal{L}\Box$.
$(S-C1)$ if $[[A]] \cap [[B]] = \emptyset$ and $y \in f(A, x)$, then $Prop(f(B, x)) = Prop(f(B, y))$, where $A \in \mathcal{L}\Box$.
$(S-C2)$ if $f(B, x) \subseteq [[A]]$ and $y \in f(A, x)$ then $Prop(f(B, x)) = Prop(f(B, y))$, where $A \in \mathcal{L}\Box$.
$(S-C3)$ if $f(B, x) \cap [[A]] \neq \emptyset$ and $y \in f(A, x)$, then $Prop(f(B, y)) \subseteq Prop(f(B, x) \cap [[A]])$, where $A \in \mathcal{L}\Box$.

We say that a formula A is true in an IBC-structure $M = \langle W, f, [[]]\rangle$ if $[[A]] = W$. We say that a formula is IBC-valid if it is true in every IBC-structure. For readability, we also use the notation $x \models A$ instead of $x \in [[A]]$.

In a IBC-structure M, we can define by means of the selection function f the *equivalence relation* $\approx_f$ on the set of worlds W as follows: for all $w, w' \in W$,

$$w \approx_f w' \text{ iff } w' \in f(\top, w).$$

The properties of $\approx_f$ being reflexive, transitive and symmetric come from the semantic conditions (REFL), (TRANS) and (EUC) of the selection function f. As a consequence of (S-BEL), all worlds in one equivalence class $[w]_{\approx_f}$ evaluate conditional formulas in the same way. Moreover, by (EUC) and (TRANS), the set $f(A, w)$ is an equivalent class in itself. We will see in the next section that each model M determines an iterated belief revision system , just by considering the equivalence classes as epistemic states and the revision operator $*$ as the canonical extension of f on the equivalence classes.

The other semantic conditions are needed to represent our postulates. From (S-UNIV), which corresponds to (U4) and (U5), and from (S-MOD) we get the property: if $[[A]] \neq \emptyset$, then $f(A, w) \neq \emptyset$. This property is needed to model the revision postulate (K5).

The semantic conditions (S-C1), (S-C2) and (S-C3) are associated with the axioms (C1), (C2) and (C3) for iterated belief revision.

The axiomatization is sound and complete with respect to the semantic properties.

Theorem 5 (Soundness) *If a formula A is a theorem of IBC then is IBC-valid.*

Theorem 6 (Completeness) *If A is IBC-valid then it is a theorem of IBC.*

4 Conditionals and Iterated Revision

In this section, we show a correspondence between iterated belief revision systems and IBC-structures.
In one direction, to each iterated belief revision system $\langle S, *, [\ ]\rangle$ corresponds an IBC-structure $\langle W, f, [[]]\rangle$ where W is a set of pairs (Ψ, w) such that Ψ is an element of S and w is a classical interpretation that satisfies the beliefs $[\Psi]$ associated to Ψ; f is defined by means of the revision operator * by saying that $f(A, (\Psi, w)) = \{(\Psi', w') : \Psi' = \Psi * A\}$; $[[p]] = \{(\Psi, w) : w \models p\}$.

In the other direction, we show that each IBC-structure $\langle W, f, [[]]\rangle$ gives rise to an iterated belief revision system. Indeed, consider the structure $\langle W/\approx_f, *_M, [\,]\rangle$, where $W/\approx_f$ is the quotient of W with respect to $\approx_f$, $*_M$ is the canonical extension of f with respect to $\approx_f$ and $[\,]$ is the function $Prop$ that associates to each set of worlds the set of propositional formulas true in all worlds of the set. This structure is a belief revision system where epistemic states are the equivalence classes of W, the belief sets associated to epistemic states are the sets of propositional formulas holding in the epistemic states and $*_M$ is the revision operator.

We say that an IBC-structure satisfies the *covering condition* if, for any A consistent, $[[A]] \neq \emptyset$.

Theorem 7 (Representation Theorem) *(1) Given a belief revision system $\langle \mathbf{S}, *, [\ ]\rangle$, there is an IBC-structure $M_* = \langle W, f, [[]]\rangle$ such that:*
for every Ψ in **S***, there exists w in W such that:*

$$B \in [\Psi * A_1 * \ldots * A_n] \textit{ iff } w \models A_1 > \ldots > A_n > B.$$

(2) Given an IBC-structure $M = \langle W, f, [[]] \rangle$ which satisfies the covering condition*, there is an iterated belief revision system $\langle \mathbf{W}/_{\approx \mathbf{f}}, *_M, [\,] \rangle$ such that:*

$\mathbf{W}/_{\approx \mathbf{f}} = \{[w]_{\approx f} : w \in W\}$;
$[w]_{\approx f} *_M A = f(A, w)$;
$[[w]_{\approx f}] = Prop([w]_{\approx f})$

and, for each $[w]_{\approx f}$ of $W/_{\approx f}$, and $A_1 \dots A_n, B \in \mathcal{L}$ ($A_1 \dots A_n$, B consistent),

$$B \in [[w]_{\approx f} *_M A_1 \dots *_M A_n] \text{ iff } w \models A_1 > \dots A_n > B.$$

The representation theorem establishes a relation between conditionals and belief revision which is reminiscent of the Ramsey Test. However, differently from the Ramsey Test, the relation stated by the representation theorem does not entail the triviality of every belief revision system, since it holds also for non-trivial ones. Here we show that our logic is not trivial according to the definition by [9].

Definition 8 A logic L is said to be *non-trivial* [9] if there are at least four formulas ϕ, ψ, χ, μ such that $\vdash_L \neg(\psi \wedge \chi)$, $\vdash_L \neg(\psi \wedge \mu)$, $\vdash_L \neg(\chi \wedge \mu)$ and $(\phi \wedge \psi)$, $(\phi \wedge \chi)$, $(\phi \wedge \mu)$ are consistent in L.

We can now prove that our logic is non-trivial. As in the proof of the following theorem we make use of Spohn's revision operator, let us recall it here.

Spohn's revision operator is a method to revise *rankings*, i.e. functions from sets of possible worlds into the class of ordinals. Rankings can be seen as a natural representation of epistemic states. Given then a set of worlds W, consider the structure $\langle \mathbf{K}, *, [\ ] \rangle$, where $\mathbf{K} = \{k : W \rightarrow Ord\}$; * is Spohn's revision operator defined as follows [6]:

$$k * A(w) = \begin{cases} k(w) - k(A) \text{ if } w \models A \\ k(w) + 1 \text{ otherwise} \end{cases}$$

and $[\ ] : \mathbf{K} \rightarrow P(\mathcal{L})$, such that $[k] = Prop\{w : k(w) = 0\}$. Since * satisfies postulates (R*1)-(R*6), (A1)-(A4), the structure $\langle \mathbf{K}, *, [\ ] \rangle$ is an iterated belief revision system.

Theorem 9 *The logic IBC is non-trivial.*

Proof. (Sketch) Consider the language L containing only the propositional variables p_1, p_2, p_3, p_4. Let $\phi = p_1; \psi = \neg p_2 \wedge \neg p_3 \wedge p_4; \chi = \neg p_3 \wedge \neg p_4 \wedge p_2; \mu = \neg p_2 \wedge \neg p_4 \wedge p_3$.
Clearly, $\vdash_{IBC} \neg(\psi \wedge \chi), \vdash_{IBC} \neg(\psi \wedge \mu)$ and $\vdash_{IBC} \neg(\chi \wedge \mu)$.
To show that $(\phi \wedge \psi), (\phi \wedge \chi)$ and $(\phi \wedge \mu)$ are consistent in IBC, we build an $IBC-$ model that satisfies them.
We proceed by building the model $< W, f, [[]] >$ as follows:

- $W = \{(k, w) : w \in 2^{\{p_1, p_2, p_3, p_4\}},\ k : 2^{\{p_1, p_2, p_3, p_4\}} \rightarrow Ord$ and $k(w) = 0\}$.
- let * be Spohn's revision operator. We define f as follows: if $A \in \mathcal{L}$, $f(A, (k, w)) = \{(k', w') \in W : k' = k * A\}$; if $\exists \phi_A \in \mathcal{L} : [[A]] = [[\phi_A]]$, $f(A, (k, w)) = f(\phi_A, (k, w))$; otherwise, $f(A, (k, w)) = \emptyset$.
- $[[p]] = \{(k, w) \in W : w \models p\}$.

This model satisfies all the semantic properties of IBC. The proof is similar to the proof of the first part of the Representation Theorem. Moreover, as W contains all pairs (k, w) such that $k(w) = 0$, for any $k : 2^{\{p_1, p_2, p_3, p_4\}} \rightarrow Ord$, it will also contain all pairs (k_1, w) for k_1 such that $k_1(w) = 0$ if $w \models p_1$, and $k_1(w) \neq 0$ otherwise. There will then be $(k_1, w_1), (k_1, w_2), (k_1, w_3) \in W$ such that $(k_1, w_1) \models \phi \wedge \psi, (k_1, w_2) \models \phi \wedge \chi, (k_1, w_3) \models \phi \wedge \mu$. Thus, $(\phi \wedge \psi), (\phi \wedge \chi)$ and $(\phi \wedge \mu)$ are consistent in IBC. $\square$

By the representation theorem this property gives an indiect[7] proof that there exists a non-trivial belief revision system.

5 Conclusions

In this paper we have presented the conditional logic IBC to capture iterated belief revision. The logic IBC provides a natural representation of epistemic states and belief sets. We have proved a representation result which establishes a mutual correspondence between iterated belief revision systems and the models of our logic. Finally, we have shown that the logic IBC is non-trivial.

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[6] This is the simplified version of Spohn's function proposed in [3]

[7] One can give a more direct proof by recalling that Spohn's operator satisfies our postulates for iterated revision and then considering an epistemic state determined by a ranking and three propositions as in the above proof.