Spatial reasoning in RCC-8 with Boolean region terms

Frank Wolter[1] and **Michael Zakharyaschev**[2]

Abstract. We extend the expressive power of the region connection calculus RCC-8 by allowing applications of the 8 binary relations of RCC-8 not only to atomic regions but also to Boolean combinations of them. It is shown that the statisfiability problem for the extended language in arbitrary topological spaces is still in NP; however, it becomes PSPACE-complete if only the Euclidean spaces $\mathbb{R}^n$, $n > 0$, are regarded as possible interpretations. In particular, in contrast to pure RCC-8, the new language is capable of distinguishing between connected and non-connected topological spaces.

1 INTRODUCTION

RCC-8 is a logical formalism intended for representing qualitative information about relationships among spatial regions in terms of 8 jointly exhaustive and pairwise disjoint basic binary predicates. Typical RCC-8 expressions are: $\mathsf{PO}(Italy, Alps)$ ('Italy and the Alps partially overlap'), $\mathsf{NTPP}(Luxemburg, EU)$ ('Luxemburg is a nontangential proper part of the EU'). RCC-8 was constructed (independently and almost simultaneously) by two parallel research streams of spatial KR&R: in the framework of geographical information systems [3] (see also [4, 2, 7]) and as an effective fragment of the much more expressive region connection calculus RCC [10] (for a study of its computational behaviour consult e.g. [8, 12, 14, 15]). The former root of RCC-8 demonstrates its practical applicability, while the latter tempts to search for more expressive and yet effective fragments.

One apparent 'deficit' of RCC-8 is that it operates only with *atomic* regions. We can't form unions ($\vee$) or intersections ($\wedge$) of regions to say, for instance, that $\mathsf{EQ}(EU, Spain \vee Italy \vee \dots)$ ('the EU consists of Spain, Italy, etc.'), $\mathsf{P}(Alps, Italy \vee France \vee \dots)$ ('the Alps are located in Italy, France, etc.'), $\mathsf{EC}(Austria, Alps \wedge Italy)$ ('Austria is externally connected to the alpine part of Italy'), and deduce from these that if $\mathsf{EC}(X, EU)$, for some country X, then $\mathsf{EC}(X, Y)$ for some country Y in the EU, or that there is a country Z such that $\mathsf{TPP}(Z, EU)$ (i.e., 'Z is a tangential proper part of the EU'). Note by the way that the last formula is a correct conclusion only if we interpret our formulas in Euclidean (or, more generally, connected) topological spaces (and if there are non-EU countries): in a discrete topological space the EU may be an open set with empty boundary. This simple observation and the result of [12], according to which every satisfiable RCC-8 formula is satisfiable in all Euclidean spaces $\mathbb{R}^n$, $n \geq 1$, show that the Boolean region terms indeed increase the expressive power of RCC-8.

The main aim of this paper is to study the computational complexity of spatial reasoning in the language of RCC-8 extended with the possibility to form Boolean combinations of regions. (As full RCC also contains region terms of this kind, the resultant language can still be regarded as a fragment of RCC.) We will show that the satisfiability problem for formulas of this language is NP-complete—that is the same as for RCC-8 formulas [15]—if arbitrary topological spaces are allowed as possible interpretations, and that it becomes PSPACE-complete if we consider only connected topological spaces, or only Euclidean ones.

[1] Institut für Informatik, Universität Leipzig, Augustus-Platz 10-11, 04109 Leipzig, Germany, email: wolter@informatik.uni-leipzig.de.
[2] School of Computer Studies, University of Leeds, Leeds LS2 9JT, England, email: mz@scs.leeds.ac.uk.

2 RCC-8

The language of RCC-8 contains individual variables $X_1, X_2, \dots$, called *region variables*, eight binary predicates DC, EC, PO, EQ, TPP, TPPi, NTPP, NTPPi, and the Boolean connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$. The well-formed formulas of this language, or RCC-8 *formuls*, are Boolean combinations of the eight predicates with region variables as their arguments.

RCC-8 formulas are often interpreted in topological spaces $\mathfrak{T} = \langle U, \mathbb{I} \rangle$, where $\mathbb{I}$ is an *interior operator* on a set U satisfying the standard Kuratowski axioms: $\mathbb{I}(X \cap Y) = \mathbb{I}X \cap \mathbb{I}Y$, $\mathbb{I}X \subseteq \mathbb{I}\mathbb{I}X$, $\mathbb{I}X \subseteq X$, $\mathbb{I}U = U$. The region variables are assumed to range over *regular closed sets* of $\mathfrak{T}$.[3] Thus an *assignment* in $\mathfrak{T}$ is a map $\mathfrak{a}$ associating with every variable X a set $\mathfrak{a}(X) \subseteq U$ such that $\mathfrak{a}(X) = \mathbb{C}\mathbb{I}\,\mathfrak{a}(X)$, where $\mathbb{C}$ is the closure operator on U dual to $\mathbb{I}$ (i.e., $\mathbb{C}Y = U - \mathbb{I}(U - Y)$). The intended meaning of the eight basic RCC-8 predicates is as follows:

$$\mathsf{DC}(X_1, X_2) \Leftrightarrow \neg\exists x\ x \in X_1 \cap X_2,$$

$$\mathsf{EC}(X_1, X_2) \Leftrightarrow (\exists x\ x \in X_1 \cap X_2) \wedge (\neg\exists x\ x \in \mathbb{I}X_1 \cap \mathbb{I}X_2),$$

$$\mathsf{PO}(X_1, X_2) \Leftrightarrow (\exists x\ x \in \mathbb{I}X_1 \cap \mathbb{I}X_2) \wedge (\exists x\ x \in \mathbb{I}X_1 \cap \neg X_2) \wedge (\exists x\ x \in \neg X_1 \cap \mathbb{I}X_2),$$

$$\mathsf{EQ}(X_1, X_2) \Leftrightarrow \forall x\ (x \in X_1 \leftrightarrow x \in X_2),$$

$$\mathsf{TPP}(X_1, X_2) \Leftrightarrow (\forall x\ x \in \neg X_1 \cup X_2) \wedge (\exists x\ x \in X_1 \cap \mathbb{C}\neg X_2) \wedge (\exists x\ x \in \neg X_1 \cap X_2),$$

$$\mathsf{NTPP}(X_1, X_2) \Leftrightarrow (\forall x\ x \in \neg X_1 \cup \mathbb{I}X_2) \wedge (\exists x\ x \in \neg X_1 \cap X_2),$$

$$\mathsf{TPPi}(X_1, X_2) \Leftrightarrow \mathsf{TPP}(X_2, X_1),$$

$$\mathsf{NTPPi}(X_1, X_2) \Leftrightarrow \mathsf{NTPP}(X_2, X_1).$$

An RCC-8 formula ϕ is said to be *satisfiable* if there exist a topological space $\mathfrak{T}$ and an assignment $\mathfrak{a}$ in it under which ϕ is true in $\mathfrak{T}$, $\mathfrak{T} \models^{\mathfrak{a}} \phi$ in symbols. Quite often in spatial representation and reasoning we are interested in satisfiability not in arbitrary topological space, but in certain specific ones, say, *connected spaces* (which are not unions of two disjoint non-empty open sets) or *Euclidean spaces* $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$ with their natural topology.

[3] It is often assumed also that the sets interpreting region variables are non-empty. In the extended language to be defined in the next section this assumption can be expressed explicitly as a spatial formula.

That the general *satisfiability problem* for RCC-8 formulas is decidable was observed by Bennett [1] who embedded RCC-8 into the bimodal (propositional) logic $\mathsf{S4}_u$—Lewis's S4 with the universal modality—using the fact that it is complete with respect to topological spaces (see also [9] for a strict proof). Here is a variant of such an embedding.

Denote by $\boldsymbol{I}$ and $\boldsymbol{C}$ the necessity and possibility operators of S4, respectively, and let $\forall$ and $\exists$ be two additional 'universal' modalities. The formulas of the resulting bimodal language $\mathcal{ML}$ are interpreted in topological spaces in the following way. Given a space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$, define a *valuation* $\mathfrak{V}$ of $\mathcal{ML}$ in $\mathfrak{T}$ as a map associating with every propositional variable p a subset $\mathfrak{V}(p)$ of U. The pair $\mathfrak{M} = \langle \mathfrak{T}, \mathfrak{V} \rangle$ is called then a *topological model* of $\mathcal{ML}$. The operators $\boldsymbol{I}$ and $\boldsymbol{C}$ are interpreted in this model as the interior and closure operators $\mathbb{I}$ and $\mathbb{C}$ of $\mathfrak{T}$, respectively, the Boolean connectives as the corresponding set-theoretic operations, and for every $\mathcal{ML}$-formula φ,

$$\mathfrak{V}(\forall\varphi) = \begin{cases} U & \text{if } \mathfrak{V}(\varphi) = U, \\ \emptyset & \text{otherwise;} \end{cases} \qquad \mathfrak{V}(\exists\varphi) = \begin{cases} U & \text{if } \mathfrak{V}(\varphi) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

The set of $\mathcal{ML}$-formulas φ that are *valid* in all topological models (in the sense that $\mathfrak{V}(\varphi) = U$) is denoted by $\mathsf{S4}_u$. Syntactically the logic $\mathsf{S4}_u$ can be defined as the fusion of S4 (with $\boldsymbol{I}$ and $\boldsymbol{C}$) and S5 (with $\forall$ and $\exists$) plus one extra axiom $\forall\varphi \to \boldsymbol{I}\varphi$. As follows from [5], $\mathsf{S4}_u$ is characterised by the class of topological spaces determined by (finite) Kripke frames for S4. Let $\mathfrak{F} = \langle W, R \rangle$ be such a frame (i.e., R is a reflexive and transitive relation, or a *quasi-order*, on W). The *topological space determined by* $\mathfrak{F}$ is the pair $\mathfrak{T}_\mathfrak{F} = \langle W, \mathbb{I}_\mathfrak{F} \rangle$ where, for every $X \subseteq W$, $\mathbb{I}_\mathfrak{F} X = \{x \in W : \forall y \in W\ (xRy \to y \in X)\}$. It is not hard to see that $\mathfrak{F}$ and $\mathfrak{T}_\mathfrak{F}$ validate precisely the same $\mathcal{ML}$-formulas.

The language of $\mathsf{S4}_u$ is expressive enough to encode the topological meaning of spatial formulas. Indeed, with every RCC-8 predicate $P(X_i, X_j)$ we can associate an $\mathcal{ML}$-formula $(P(X_i, X_j))^*$ defined by taking:

$(\mathsf{DC}(X_i, X_j))^* = \neg\exists(p_i \wedge p_j)$,
$(\mathsf{EC}(X_i, X_j))^* = \exists(p_i \wedge p_j) \wedge \neg\exists(\boldsymbol{I}p_i \wedge \boldsymbol{I}p_j)$,
$(\mathsf{PO}(X_i, X_j))^* = \exists(\boldsymbol{I}p_i \wedge \boldsymbol{I}p_j) \wedge \exists(\boldsymbol{I}p_i \wedge \neg p_j) \wedge \exists(\neg p_i \wedge \boldsymbol{I}p_j)$,
$(\mathsf{EQ}(X_i, X_j))^* = \forall(p_i \leftrightarrow p_j)$,
$(\mathsf{TPP}(X_i, X_j))^* = \forall(\neg p_i \vee p_j) \wedge \exists(p_i \wedge \boldsymbol{C}\neg p_j) \wedge \exists(\neg p_i \wedge p_j)$,
$(\mathsf{NTPP}(X_i, X_j))^* = \forall(\neg p_i \vee \boldsymbol{I}p_j) \wedge \exists(\neg p_i \wedge p_j)$.

Now, given an RCC-8 formula ϕ, denote by ϕ^* the result of replacing all occurrences of the RCC-8 predicates $P(X_i, X_j)$ in ϕ by the corresponding $\mathcal{ML}$-formulas $(P(X_i, X_j))^*$. And then we put

$$\phi^\dagger = \phi^* \wedge \bigwedge_{X_i \in var\phi} (p_i \leftrightarrow \boldsymbol{CI}p_i), \qquad (1)$$

where $var\phi$ is the set of region variables in ϕ. (The last conjunct says that the variables in $\phi^\dagger$ are interpreted as regular closed sets).

Theorem 1 (Bennett) *An* RCC-8 *formula ϕ is satisfiable iff $\phi^\dagger$ is satisfiable in the topological space $\mathfrak{T}_\mathfrak{F}$ determined by some finite quasi-order $\mathfrak{F}$.*

This theorem reduces the satisfiability problem for RCC-8 formulas to the satisfiability problem for $\mathcal{ML}$-formulas in Kripke frames for $\mathsf{S4}_u$, which is known to be decidable [5]. Renz [12] showed that actually the satisfiability problem for RCC-8 formulas is NP-complete; Renz and Nebel [13, 15] described all maximal tractable fragments of RCC-8.

3 RCC-8 WITH REGION TERMS

Denote by BRCC-8 the extension of RCC-8 which allows the use of Boolean combinations of region variables as arguments of RCC-8 predicates. Such combinations are called *region terms*. Their semantical meaning is defined as follows (cf. [6]). Given a topological space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$, an assignment $\mathfrak{a}$ in it and region terms t, t', we put

- $\mathfrak{a}(t \vee t') = \mathbb{CI}\,(\mathfrak{a}(t) \cup \mathfrak{a}(t')) = \mathfrak{a}(t) \cup \mathfrak{a}(t')$,
- $\mathfrak{a}(t \wedge t') = \mathbb{CI}\,(\mathfrak{a}(t) \cap \mathfrak{a}(t'))$,
- $\mathfrak{a}(\neg t) = \mathbb{CI}\,(U - \mathfrak{a}(t)) = \mathbb{C}(U - \mathfrak{a}(t))$.

Thus every region term is interpreted as a regular closed set of $\mathfrak{T}$. Note that $\mathfrak{a}(X \wedge \neg X) = \emptyset$ and $\mathfrak{a}(X \vee \neg X) = U$ for any $\mathfrak{a}$ and $\mathfrak{T}$. We denote the terms $X \wedge \neg X$ and $X \vee \neg X$ by $\bot$ and $\top$, respectively. The constraint $\neg\mathsf{EQ}(X, \bot)$ asserts that X is a non-empty region.

Our aim in this section is to show that the satisfiability problem for BRCC-8 formulas is decidable in NP. To this end we extend the translation † of the previous section to the region terms. Given such a term t, define an $\mathcal{ML}$-formula t^* by taking

$$X_i^* = p_i, \qquad (\neg t)^* = \boldsymbol{CI}\neg t^*,$$
$$(t_1 \vee t_2)^* = \boldsymbol{CI}(t_1^* \vee t_2^*), \qquad (t_1 \wedge t_2)^* = \boldsymbol{CI}(t_1^* \wedge t_2^*).$$

For every BRCC-8 predicate $P(t_1, t_2)$ we put

$$(P(t_1, t_2))^* = (P(X_1, X_2))^*\{t_1^*/p_1, t_2^*/p_2\}$$

and define the *modal translation* $\phi^\dagger$ of a BRCC-8 formula ϕ as before by (1). It should be clear that Bennett's theorem holds for BRCC-8 formulas as well.

The modal translations of BRCC-8 formulas form a rather special fragment of $\mathcal{ML}$. For instance, Renz [12] showed that an RCC-8 formula ϕ is satisfiable iff $\phi^\dagger$ is satisfiable in a Kripke model based on an S4-frame of depth ≤ 1 and width ≤ 2, which means that the frame contains no chain of more than 2 distinct points, and no point has more than 2 distinct proper successors. It turns out that this result can be generalised to BRCC-8 formulas. To prove this, we require a number of definitions.

An $\mathcal{ML}$-formula is a $\boldsymbol{CI}$*-term* if it can be obtained from some Boolean formula χ (without modal operators) by prefixing $\boldsymbol{CI}$ to every subformula of χ. A $\boldsymbol{CI}$-term prefixed by a string of $\neg$, $\boldsymbol{I}$, and $\boldsymbol{C}$ is called a *general* $\boldsymbol{CI}$*-term*. (It is easy to see that every general $\boldsymbol{CI}$-term is equivalent in $\mathsf{S4}_u$ to a formula of the form χ, $\neg\chi$, $\boldsymbol{I}\chi$, $\neg\boldsymbol{I}\chi$, or $\boldsymbol{I}\neg\chi$, where χ is a $\boldsymbol{CI}$-term.) By a $\boldsymbol{CI}$*-formula* we mean an $\mathcal{ML}$-formula composed from formulas of the form $\exists\psi$ and $\forall\psi$, where ψ is a Boolean combination of general $\boldsymbol{CI}$-terms, using only Boolean connectives.

It easily follows from the given definitions that the modal translation of any BRCC-8 formula is equivalent in $\mathsf{S4}_u$ to a $\boldsymbol{CI}$-formula. We now show that all $\boldsymbol{CI}$-formulas satisfiable in topological models can be satisfied in Kripke $\mathcal{ML}$-models of a rather simple form.

A partial order $\langle V, S \rangle$ is of *depth* ≤ 1 iff V can be represented as the disjoint union of two sets, V_1 and V_0, in such a way that S is the reflexive closure of a subset of $V_1 \times V_0$. The points in V_i are said to be of *depth* i.

Lemma 2 *Every satisfiable $\boldsymbol{CI}$-formula φ can be satisfied in a Kripke model based on a partial order of depth ≤ 1.*

Proof As $\mathsf{S4}_u$ has the finite model property, φ is satisfied in a finite Kripke model $\mathfrak{M} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ based on a quasi-order $\mathfrak{G} = \langle W, R \rangle$. Define a partial order $\mathfrak{F} = \langle V, S \rangle$ by taking $V = V_0 \cup V_1$, where

$$V_0 = \{x \in W : \neg\exists y \in W\ (xRy \wedge \neg yRx)\}, \quad V_1 = W - V_0,$$

and taking S to be the reflexive closure of $R \cap (V_1 \times V_0)$. In other words, $\mathfrak{F}$ has the same set of worlds as $\mathfrak{G}$, but only those arrows from the latter that lead to points in final clusters (arrows within these clusters are omitted). Let $\mathfrak{V} = \mathfrak{U}$ and $\mathfrak{K} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. Then for every $\boldsymbol{CI}$-formula ψ and every $u \in V$, we have $u \models_{\mathfrak{K}} \psi$ iff $u \models_{\mathfrak{M}} \psi$. (An inductive proof can be found in the full paper at *http://www.informatik.uni-leipzig.de/~wolter*.) □

A partial order of depth ≤ 1 and width ≤ 2 is called a *quasisaw* and a Kripke model based on a quasisaw is called a *quasisaw model*.

Lemma 3 *A* BRCC-8 *formula* ϕ *is satisfiable iff* $\phi^\dagger$ *is satisfiable in a quasisaw model.*

Proof Only $(\Rightarrow)$ needs a proof. By Lemma 2, $\phi^\dagger$ is satisfiable in a Kripke model $\mathfrak{K} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ such that $\mathfrak{F} = \langle W, R \rangle$ is of depth ≤ 1. We can assume also that every point of depth 1 in $\mathfrak{F}$ has at least two proper successors.

Let Π be the set of all pairs $\{x, y\}$ of distinct points in $\mathfrak{F}$ of depth 0 for which there is a $z \in W$ with zRx and zRy. For each $\{x, y\} \in \Pi$ we take a fresh point $u_{x,y}$ and define a new frame $\mathfrak{F}' = \langle W', R' \rangle$ in which $W' = \{w \in W : dp(w) = 0\} \cup \{u_{x,y} : \{x, y\} \in \Pi\}$, and R' is the reflexive closure of $\{(u_{x,y}, x) : \{x, y\} \in \Pi\}$. Clearly, $\mathfrak{F}'$ is a quasisaw. Define a valuation $\mathfrak{V}'$ in $\mathfrak{F}' = \langle W', R' \rangle$ by taking, for every variable p, $x \in \mathfrak{V}'(p)$ iff there is $y \in W'$ of depth 0 such that $xR'y$ and $y \in \mathfrak{V}(p)$. It can be proved by induction (consult the full paper) that ϕ is satisfied in $\mathfrak{K}' = \langle \mathfrak{F}', \mathfrak{V}' \rangle$. □

It is easy to see that in the constructed model $\mathfrak{K}'$ we actually need $\leq 3\ell(\phi)$ points—to satisfy the subformulas $\exists\psi$ of $\phi^\dagger$ that hold in $\mathfrak{K}'$—and, of course, their successors, i.e., $\leq 9\ell(\phi)$ points in total, where $\ell(\phi)$ is the length of ϕ. It follows that there is a nondeterministic polynomial time algorithm for checking satisfiability of BRCC-8 formulas. Thus we obtain:

Theorem 4 *The satisfiability problem for* BRCC-8 *formulas in arbitrary topological spaces is NP-complete.*

4 SATISFIABILITY IN EUCLIDEAN SPACES

As was shown by Renz [12], for RCC-8 formulas satisfiability in arbitrary spaces coincides with satisfiability in $\mathbb{R}$, and so in $\mathbb{R}^n$ for any $n > 0$. However, this does not hold for BRCC-8 formulas.

Example 5 Consider the conjunction ϕ of the BRCC-8 formulas: $\mathsf{EQ}(X_1 \vee X_2, Y)$, $\mathsf{NTPP}(X_1, Y)$, $\mathsf{NTPP}(X_2, Y)$, $\neg\mathsf{EQ}(Y, \top)$, $\neg\mathsf{EQ}(X, \bot)$, where X ranges over $\{X_1, X_2, Y\}$. Clearly, ϕ can be satisfied in the topological space consisting of three points and having the identical interior operator. Suppose now that ϕ holds in some space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$. Then the region $X_1 \vee X_2$ is closed and included in the interior of Y. On the other hand, it coincides with Y. Hence Y is both closed and open. It follows that U is the union of two disjoint non-empty open sets, Y and $U - Y$, and so $\mathfrak{T}$ is not connected. Thus ϕ is not satisfiable in $\mathbb{R}^n$ for any $n \geq 1$. (It follows in particular that $\mathsf{S4}_u$ is not complete with respect to the class of connected spaces. Note however that S4 is sound and complete with respect to $\mathbb{R}$; see e.g. [11].)

In this section we show that the satisfiability problem for BRCC-8 formulas in $\mathbb{R}^n$, $n \geq 1$, is still decidable. However, its computational complexity grows up to PSPACE.

Say that a frame $\mathfrak{F} = \langle W, R \rangle$ is *connected* if for any two points $x, y \in W$ we have $x(R \cup R^{-1})^* y$, where $(R \cup R^{-1})^*$ is the transitive closure of the relation $R \cup R^{-1}$. In other words, if we depict $\mathfrak{F}$ as a (nondirected) graph $G_{\mathfrak{F}}$ whose nodes are points in W and edges are pairs (x, y) such that either xRy or yRx, then $G_{\mathfrak{F}}$ is connected.

Lemma 6 *Every* $\mathcal{ML}$*-formula satisfiable in a connected topological space is satisfiable in a model based on a finite connected frame.*

Proof Suppose an $\mathcal{ML}$-formula φ is satisfied in a connected topological space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ under a valuation $\mathfrak{V}$. Denote by $sub\varphi$ the set of subformulas of φ and define an equivalence relation $\sim$ on U by taking $v \sim w$ iff for every $\psi \in sub\varphi$ we have $v \in \mathfrak{V}(\psi)$ iff $w \in \mathfrak{V}(\psi)$. Let $W = \{[v] : v \in U\}$, where $[v] = \{w : w \sim v\}$. Define a binary relation R on W by taking $[v]R[w]$ iff, for every $\boldsymbol{I}\psi \in sub\varphi$, we have $w \in \mathfrak{V}(\boldsymbol{I}\psi)$ whenever $v \in \mathfrak{V}(\boldsymbol{I}\psi)$. Clearly, R is reflexive and transitive, i.e., $\mathfrak{F} = \langle W, R \rangle$ is a finite quasi-order. Let us show that $\mathfrak{F}$ is connected. Suppose otherwise. Then there are $[v]$ and $[w]$ in W such that $[v](R \cup R^{-1})^*[w]$ does not hold. Put

$$C_v = \{[u] \in W : [v](R \cup R^{-1})^*[u]\}.$$

According to our assumption, neither C_v nor $W - C_v$ is empty. For each pair $[u] \in C_v$, $[w] \in W - C_v$ select a formula $\boldsymbol{I}\alpha_{[u],[w]} \in sub\varphi$ such that $u \in \mathfrak{V}(\boldsymbol{I}\alpha_{[u],[w]})$, but $w \notin \mathfrak{V}(\boldsymbol{I}\alpha_{[u],[w]})$. This can be done because $[u]R[w]$ does not hold. And since $[w]R[u]$ does not hold either, we can choose a formula $\boldsymbol{I}\beta_{[w],[u]} \in sub\varphi$ such that $w \in \mathfrak{V}(\boldsymbol{I}\beta_{[w],[u]})$, but $u \notin \mathfrak{V}(\boldsymbol{I}\beta_{[w],[u]})$. Let

$$\alpha = \bigvee_{x \in C_v} \bigwedge_{y \in W - C_v} \boldsymbol{I}\alpha_{x,y}, \qquad \beta = \bigvee_{x \in W - C_v} \bigwedge_{y \in C_v} \boldsymbol{I}\beta_{x,y}.$$

It is easy to see that for every $u \in U$ we have

- $u \in \mathfrak{V}(\alpha)$ iff $u \in \bigcup C_v$, and $u \in \mathfrak{V}(\beta)$ iff $u \in U - \bigcup C_v$.

Hence, both $\bigcup C_v$ and $U - \bigcup C_v$ are open and nonempty, contrary to $\mathfrak{T}$ being connected. It remains to prove that φ is satisfied in $\mathfrak{F}$. Define a valuation $\mathfrak{V}'$ in $\mathfrak{F}$ by taking $\mathfrak{V}'(p) = \{[v] : v \in \mathfrak{V}(p)\}$. By induction on the construction of $\psi \in sub\varphi$ we show that $v \in \mathfrak{V}(\psi)$ iff $[v] \in \mathfrak{V}'(\psi)$. The basis of induction and the cases of the Booleans and universal modalities are trivial. Suppose $\psi = \boldsymbol{I}\chi$. The implication '$v \in \mathfrak{V}(\psi) \Rightarrow [v] \in \mathfrak{V}'(\psi)$' follows directly from the definition of R. Let us prove the converse. Assume that $[v] \in \mathfrak{V}'(\psi)$, but $v \notin \mathfrak{V}(\psi)$. As $[v] \in \mathfrak{V}'(\chi)$, we have by IH $v \in \mathfrak{V}(\chi)$. Let $\boldsymbol{I}\gamma_1, \ldots, \boldsymbol{I}\gamma_n$ be all subformulas of φ starting with $\boldsymbol{I}$ and such that $v \in \mathfrak{V}(\boldsymbol{I}\gamma_i)$, $i = 1, \ldots, n$. If such formulas do not exist, then $[v]R[w]$ for every $w \in U$, and so by IH $\mathfrak{V}(\chi) = U = \mathfrak{V}(\boldsymbol{I}\chi)$, which is a contradiction. So we may assume that $n > 0$. Let γ be the conjunction of all $\boldsymbol{I}\gamma_i$. Note that $\mathfrak{V}(\gamma) \not\subseteq \mathfrak{V}(\chi)$, for otherwise we would have $\mathfrak{V}(\boldsymbol{I}\gamma_1) \cap \cdots \cap \mathfrak{V}(\boldsymbol{I}\gamma_n) \subseteq \mathfrak{V}(\boldsymbol{I}\chi)$, contrary to $v \notin \mathfrak{V}(\psi)$. So there is a point $w \in \mathfrak{V}(\gamma) - \mathfrak{V}(\chi)$. By the choice of γ, we have $[v]R[w]$ and, by IH, $[w] \notin \mathfrak{V}'(\chi)$, contrary to $[v] \in \mathfrak{V}'(\boldsymbol{I}\chi)$. □

A connected quasisaw will be called a *saw*.

Lemma 7 *If a* BRCC-8 *formula* ϕ *is satisfiable in a connected topological space, then* $\phi^\dagger$ *is satisfiable in a finite saw model.*

Proof By Lemma 6, without loss of generality we may assume that $\phi^\dagger$ is satisfied in a finite connected quasi-order under some valuation $\mathfrak{V}$. In the same way as in the proof of Lemma 2 we construct a partial order $\mathfrak{F} = \langle W, R \rangle$ of depth ≤ 1 satisfying $\phi^\dagger$ under $\mathfrak{V}$. It should be clear that $\mathfrak{F}$ is connected. We can assume also that every point of depth 1 in $\mathfrak{F}$ has at least two proper successors. Now in precisely the same way as in the proof of Lemma 3 we construct the model $\mathfrak{K}' = \langle \mathfrak{F}', \mathfrak{V}' \rangle$ satisfying $\phi^\dagger$. Since $\mathfrak{F}$ is connected, $\mathfrak{F}'$ must be a saw. □

Theorem 8 *A* BRCC-8 *formula* ϕ *is satisfiable in* $\mathbb{R}$ *iff* $\phi^\dagger$ *is satisfiable in a finite saw of size* $\leq 2^{c\cdot\ell(\phi)}$, $c = const$.

Proof The implication ($\Rightarrow$) follows from the two preceding lemmas. Let us prove the converse. Every finite saw model for $\phi^\dagger$ can clearly be transformed into a model $\mathfrak{M} = \langle\mathfrak{F},\mathfrak{V}\rangle$ satisfying $\phi^\dagger$ and based on the frame $\mathfrak{F} = \langle\mathbb{Z}, R\rangle$ such that xRy iff $x = y$ or there exists $n \in \mathbb{Z}$ with $x = 2n$ and $y \in \{2n-1, 2n+1\}$. Now define a valuation $\mathfrak{V}'$ in $\mathbb{R}$ by taking

$$\mathfrak{V}'(p) = \bigcup_{2n+1\in\mathfrak{V}(p)} [2n, 2n+2]$$

for all propositional variables p. It is not hard to check that $\phi^\dagger$ is satisfied in the topological model $\langle\mathbb{R}, \mathfrak{V}'\rangle$. □

As a consequence of Theorems 8 and 4 we obtain:

Theorem 9 *The satisfiability problem for* BRCC-8 *formulas in* $\mathbb{R}$ *is decidable in PSPACE.*

Proof By Savitch's theorem, it suffices to present a nondeterministic polynomial space algorithm. It consists of two parts. The first one is the nondeterministic polynomial time algorithm provided by Theorem 4. It guesses a quasisaw model $\mathfrak{M}$ satisfying a given formula ϕ and containing $m \leq 9\cdot\ell(\phi)$ points, together with the set

$$\Xi = \{\neg\exists\psi \in sub\phi^\dagger : \models_{\mathfrak{M}} \neg\exists\psi\} \cup \{p_i \leftrightarrow \boldsymbol{CI}p_i : X_i \in var\phi\}$$

and the set Π of all pairs of points of depth 0 in $\mathfrak{M}$ that are not connected by the quasisaw. The second algorithm checks whether a pair $(x,y) \in \Pi$ can be connected by a saw model with $\leq 2^{c\cdot\ell(\phi)}$ points validating Ξ. To this end we guess a number $n \leq 2^{c\cdot\ell(\phi)}$ and represent it in binary (which requires polynomial space). Set $i = 1$, $x_i = x$ and $x_n = y$. If $i + 2 = n$ then we guess one point x_{i+1} together with a valuation of $\phi^\dagger$'s variables in it, and check whether all formulas in Ξ are true at x_{i+1} provided that x_i and x_{i+2} are the only immediate successors of x_{i+1}. If this is the case, then (x,y) can be connected; we delete it from Π and check the remaining pairs. Now, if $i+2 < n$ then we guess two points x_{i+1} and x_{i+2} together with a valuation of $\phi^\dagger$'s variables in these points, and check whether all formulas in Ξ are true at x_{i+1} and x_{i+2} provided that x_i and x_{i+2} are the only immediate successors of x_{i+1}. If this is indeed the case then we proceed to considering x_{i+2} and forget everything about x_j, for $j < i+2$, thus remaining within polynomial space. □

Moreover, it turns out that the established upper bound cannot be made smaller.

Theorem 10 *The satisfiability problem for* BRCC-8 *formulas in* $\mathbb{R}$ *is PSPACE-complete.*

Proof Let L be a language in the alphabet $\{0,1\}$ and let L be in PSPACE. Our aim is to show that there is a translation f of words in $\{0,1\}$ into the language of BRCC-8 such that

- for every $e \in \{0,1\}^*$, $e \in L$ iff $f(e)$ is satisfiable in $\mathbb{R}$, and
- $f(e)$ is computable in time polynomial in $|e|$, where $|e|$ is the length of e.

Since L is in PSPACE, there is a one-tape right-infinite Turing machine $\mathfrak{A}$ which, starting from an arbitrary $e \in \{0,1\}^*$ in the initial state q_1 reaches the final state q_0 iff $e \in L$, and while working the head of the machine never moves to the right of cell $\mathcal{P}(|e|)$, for some fixed polynomial $\mathcal{P}$. Thus, for a word e of length n, the working zone of $\mathfrak{A}$ consists of the cells $0,\dots,k = \mathcal{P}(n)$; the cells $k+1,\dots$ are empty. Let $q_0,\dots,q_m$ be the states of $\mathfrak{A}$. We will assume that instructions of $\mathfrak{A}$ are of the following three types:

$$q_i 1^\sigma \Rightarrow Rq_j, \quad q_i 1^\sigma \Rightarrow Lq_j, \quad q_i 1^\sigma \Rightarrow q_j 1^\tau.$$

Here $\sigma,\tau \in \{0,1\}$, $X^\sigma = X$ if $\sigma = 1$ and $X^\sigma = \neg X$ otherwise ($\neg 1 = 0$, $\neg 0 = 1$). The meaning of these instructions is as follows: if $\mathfrak{A}$ is in state q_i and its head reads 1^σ, then $\mathfrak{A}$ goes to state q_j and moves its head one cell to the right (the first instruction), one cell to the left (the second one), or does not move the head, but writes 1^τ in the active cell.[4]

With e and $\mathfrak{A}$ we associate the following region variables:

- $X_0,\dots,X_k$ (to represent the cells of $\mathfrak{A}$);
- $Y_0,\dots,Y_m$ (to represent the states of $\mathfrak{A}$);
- $Z_0,\dots,Z_k$ (to represent the position of the head of $\mathfrak{A}$).

With every instruction $q_i 1^\sigma \Rightarrow Rq_j$ in $\mathfrak{A}$ we associate the pairs

$$\{Y_i \wedge X_l^\sigma \wedge X_{l+1}^\tau \wedge Z_l, Y_j \wedge X_l^\sigma \wedge X_{l+1}^\tau \wedge Z_{l+1}\}, \qquad (2)$$

where $0 \leq l < k$, $\tau \in \{0,1\}$; with every instruction $q_i 1^\sigma \Rightarrow Lq_j$ in $\mathfrak{A}$ we associate the pairs

$$\{Y_i \wedge X_{l-1}^\tau \wedge X_l^\sigma \wedge Z_l, Y_j \wedge X_{l-1}^\tau \wedge X_l^\sigma \wedge Z_{l-1}\}, \qquad (3)$$

where $0 < l \leq k$, $\tau \in \{0,1\}$; and with instructions $q_i 1^\sigma \Rightarrow q_j 1^\tau$ the pairs

$$\{Y_i \wedge X_l^\sigma \wedge Z_l, Y_j \wedge X_l^\tau \wedge Z_l\}, \quad 0 \leq l \leq k. \qquad (4)$$

Denote by Ξ the set of all pairs of the form (2)–(4) having distinct components and different from the pairs associated with the instructions in $\mathfrak{A}$. Clearly, $|\Xi|$ is polynomial in $|e|$.

Now, for every $\{t,t'\} \in \Xi$ we take the formula

$$\mathsf{DC}(t,t') \qquad (5)$$

and define $f(e)$ as the conjunction of all these formulas and the following ones:

$$\neg\mathsf{EQ}(X_0^{\sigma_0} \wedge\cdots\wedge X_k^{\sigma_k} \wedge Y_1 \wedge Z_0, \bot),\ e = (\sigma_0,\dots,\sigma_k), \qquad (6)$$
$$\neg\mathsf{EQ}(Y_0,\bot), \qquad (7)$$
$$\mathsf{EQ}(\top, Y_0 \vee\cdots\vee Y_m), \qquad (8)$$
$$\mathsf{DC}(Y_i,Y_j) \vee \mathsf{EC}(Y_i,Y_j),\ 0 \leq i \neq j \leq m, \qquad (9)$$
$$\mathsf{EQ}(\top, Z_0 \vee\cdots\vee Z_k), \qquad (10)$$
$$\mathsf{DC}(Z_i,Z_j) \vee \mathsf{EC}(Z_i,Z_j),\ 0 \leq i \neq j \leq k, \qquad (11)$$
$$\mathsf{DC}(Z_i,Z_j),\ i = 0,\dots,k,\ j \neq i-1,i,i+1, \qquad (12)$$
$$\mathsf{DC}(X_i \wedge Z_j, \neg X_i),\ i \neq j, \qquad (13)$$
$$\mathsf{DC}(\neg X_i \wedge Z_j, X_i),\ i \neq j. \qquad (14)$$

It is readily seen that $f(e)$ is computable in time polynomial in $|e|$.

Suppose $f(e)$ is satisfied in $\mathbb{R}$. Then by Theorem 8, $f(e)$ is satisfied in the topological space determined by a saw $\mathfrak{F} = \langle W,R\rangle$. Let x,y,z be three distinct points in W such that zRx and zRy, and let

$$x \models X_0^{\rho_0} \wedge\cdots\wedge X_k^{\rho_k} \wedge Y_i \wedge Z_l, \quad y \models X_0^{\tau_0} \wedge\cdots\wedge X_k^{\tau_k} \wedge Y_j \wedge Z_{l'}.$$

[4] More frequently used instructions of the form $I : q_i 1^\sigma \Rightarrow Dq_j 1^\tau$ can be simulated by two our instructions: $q_i 1^\sigma \Rightarrow q^I 1^\tau$, $q^I 1^\tau \Rightarrow Dq_j$, where q^I is a new state corresponding to I.

In view of (12), $|l - l'| \in \{0, 1\}$. And by (13), (14), we have: if $l \neq l'$ then $\rho_r = \tau_r$ for all r, otherwise $\rho_r = \tau_r$ for all $r \neq l$. It follows by (5) that $\mathfrak{A}$ contains either one of the instructions

$$q_i 1^{\rho_l} \Rightarrow R q_j \qquad (\rho_l = \tau_l,\ l' = l + 1),$$
$$q_i 1^{\rho_l} \Rightarrow L q_j \qquad (\rho_l = \tau_l,\ l' = l - 1),$$
$$q_i 1^{\rho_l} \Rightarrow q_j 1^{\tau_l} \qquad (l' = l)$$

(it transforms the configuration corresponding to x into the configuration corresponding to y) or one of the instructions

$$q_j 1^{\tau_{l'}} \Rightarrow R q_i \qquad (\rho_l = \tau_l,\ l = l' + 1),$$
$$q_j 1^{\tau_{l'}} \Rightarrow L q_i \qquad (\rho_l = \tau_l,\ l = l' - 1),$$
$$q_j 1^{\tau_{l'}} \Rightarrow q_i 1^{\rho_l} \qquad (l' = l)$$

(it transforms the configuration corresponding to y into the configuration corresponding to x) or one instruction from either of these sets. We call this instruction(s) the *instruction(s) for* $\{x, y\}$. In view of (9) and (12), they are uniquely determined by $\{x, y\}$ (e.g., if $x \models Y_i \wedge Y_j$ then $i = j$).

As $f(e)$ is satisfied in our model and in view of (6), (7) and (10), we have points x and y of depth 0 in $\mathfrak{F}$ such that

$$x \models X_0^{\sigma_0} \wedge \cdots \wedge X_k^{\sigma_k} \wedge Y_1 \wedge Z_0, \quad y \models X_0^{\tau_0} \wedge \cdots \wedge X_k^{\tau_k} \wedge Y_0 \wedge Z_{l'},$$

for some τ_i and l'. Since $\mathfrak{F}$ is connected, we can choose a minimal number of points $x_1, \ldots, x_r$ such that $x_0 = x$, $x_r = y$ and for every i, $1 \leq i < r$, there is $y_i \in W$ with $y_i R x_i$, $y_i R x_{i+1}$. Our aim is to show that $\mathfrak{A}$, having started from the tape $\sigma_0, \ldots, \sigma_k$, comes to a stop (i.e., reaches q_0) on the tape $\tau_0, \ldots, \tau_k$. If this is the case then $e \in L$.

Without loss of generality we may assume that no x_i validates Y_0 if $i < r$, and that no x_i, x_j can be connected directly whenever $j > i + 1$ (i.e., we cannot add a point y to W so that $y R x_i$ and $y R x_j$ without violating the constraints (5)–(14)). Consider now the instruction(s) for some pair $\{x_s, x_{s+1}\}$, $0 \leq s < r$. We claim that there is only one such instruction, and it transforms the configuration corresponding to x_s into the configuration corresponding to x_{s+1}. Indeed, suppose that $\{x_s, x_{s+1}\}$ is the last pair for which this is not the case. Since no instruction may contain q_0 in its left-hand part, $s < r - 1$. And since no two instructions of $\mathfrak{A}$ may have the same left-hand side, the configurations corresponding to x_s and x_{s+2} coincide. So either x_s is 'terminal' or it can be connected directly with x_{s+3}, contrary to the minimality of r. It follows that $e \in L$.

Conversely, suppose $e = (\sigma_0, \ldots, \sigma_k)$ is in L. Then, having started from $(\sigma_0, \ldots, \sigma_k)$ in state q_1, in $s \leq 2^k$ steps $\mathfrak{A}$ will reach the halt state q_0 without moving its head to the right of cell $k = \mathcal{P}(|e|)$. Denote by $(\sigma_0^n, \ldots, \sigma_k^n)$ the state of the tape at step n, by $q(n)$ the number of the state of $\mathfrak{A}$ at step n, and by $h(n)$ the number of the active cell at step n. Construct a frame $\mathfrak{F} = \langle W, R \rangle$ by taking

$$W = \{x_0, \ldots, x_s, y_0, \ldots, y_{s-1}\},$$
$$R = \{\langle y_i, x_i \rangle, \langle y_i, x_{i+1} \rangle, \langle x, x \rangle : x \in W,\ i = 0, \ldots, s - 1\}.$$

Define a valuation in $\mathfrak{F}$ as follows:

$$x_i \models X_j \quad \text{iff} \quad \sigma_j^i = 1,$$
$$x_i \models Y_j \quad \text{iff} \quad j = q(i),$$
$$x_i \models Z_j \quad \text{iff} \quad j = h(i),$$
$$y_i \models X \quad \text{iff} \quad x_i \models X \text{ or } x_{i+1} \models X,$$

for any region variable X in $f(e)$. It is readily checked that the modal translation of $f(e)$ is satisfied in the resultant saw model. □

5 CONCLUSION

We have determined the computational complexity of the satisfiability problem for RCC-8 formulas with Boolean terms in both arbitrary topological spaces and Euclidean ones. This research can be regarded as a first step towards understanding effective extensions of RCC-8 included in the undecidable RCC. Two obvious open problems in this direction are: (1) What happens if we interpret region variables only by connected (regular closed) sets of arbitrary or Euclidean topological spaces? (2) Are there interesting tractable fragments of BRCC-8 which are not fragments of RCC-8?

ACKNOWLEDGEMENTS

We would like to thank A. Cohn and A. Kurucz for comments on the first draft of this paper.

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