Characterizing general preferential entailments

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Abstract. A preferential entailment is defined by a binary relation, or "preference relation". This relation can be either among interpretations or among sets of interpretations. The relation can be also among "states" which are "copies of interpretations", or "copies of sets of interpretations". This provides four kinds of preferential entailments. The paper deals mainly with propositional logic, however this work applies also to the first order case and indications are given in order to describe the situation in first order logic. What we do here is to provide a characterization result for the most general version described above, and to compare with the known characterizations of the "simplest" versions. It appears that the apparently most complicated notion possesses by far the simplest characterization result. A by-product of our results is that the definition can be simplified without loss of generality: we can define directly the relation among sets of interpretations, eliminating the need for "states" in this case. Thus only three kinds of "preferential entailment" remain.

1 Introduction

The notion of preferential entailment has shown to be very useful in knowledge representation, when dealing with some aspects of common sense reasoning such as implicit knowledge or rules with exceptions. Preferential entailments are particular cases of inference operations, and various kinds exist in the literature. The general idea is as follows: we are given some amount of certain knowledge, represented as a set of logical formulas. This set, equivalent to a logical theory, can be associated with various kinds of objects. For instance, it can be associated with its set of models, or equivalently in the propositional case with the set of the complete theories which entail the given formulas. More generally, it can also be associated with the set of theories (not necessarily complete) which entail the given formulas. Then we are given a binary relation among these objects associated with our knowledge, and we keep only the objects which are "preferred" for this relation, meaning the objects which are minimal for this relation, among the objects associated with our certain knowledge. We get a stronger set of formulas, which are considered as deduced "by default", meaning that in the absence of other information, we conclude that we get also all the formulas associated with this reduced set of objects. This allows to reason in a non monotonic way, as augmenting the certain knowledge may invalidate some conclusions previously made by defaults, because some new objects which were not minimal may be minimal in the smaller set associated with the new certain knowledge. We can even allow more flexibility by considering copies of models, or copies of theories, instead of just models or theories, defining the relation among these sets of "copies". We get then four kinds of preferential entailments. We show that no real additional flexibility is obtained by replacing the theories by copies of theories, which leaves three kinds only. We provide also a characterization result, in terms of a very simple syntactical property, for the "most complicated" version. To our knowledge, these results are new: they were known for particular cases, but not for the general case.

In section 2 we introduce the notations used in the text. We work in propositional logic, which is rich enough for studying the problems addressed here. In sections 3 and 4, we give a few reminders about the simplest kinds of preferential entailments. Section 5 gives our results for the most general version of preferential entailment, with examples, hints for the first order case, and a comparison with the already known results about the simplest kinds of preferential entailment.

2 Notations and framework

• $\mathbf{L},\ V,\ \varphi,\ \mathcal{T}$: We work in a propositional language $\mathbf{L}$. As usual, $\mathbf{L}$ also denotes the set of all the formulas. $V(\mathbf{L})$, the vocabulary of $\mathbf{L}$, denotes a set of *propositional symbols*. Letters φ, ψ denote formulas in $\mathbf{L}$. A *formula* will be assimilated to its equivalence class. Letters such as $\mathcal{T}$ or $\mathcal{C}$ denote sets of formulas.

• $\mathbf{M},\ \mu,\ \mathcal{P}(E),\ \mu \models \cdots$: Letters μ, ν denote *interpretations for* $\mathbf{L}$. An interpretation is identified with the subset of $V(\mathbf{L})$ that it satisfies. The writings $\mu \models \varphi$ and $\mu \models \mathcal{T}$ are defined classically. As usual, for any set E, $\mathcal{P}(E)$ denotes the set of all the subsets of E. The set $\mathcal{P}(V(\mathbf{L}))$ of all the interpretations for $\mathbf{L}$ is denoted by $\mathbf{M}$. A *model* of $\mathcal{T}$ is an interpretation μ such that $\mu \models \mathcal{T}$. The sets of the models of $\mathcal{T}$ and φ are denoted by $\mathbf{M}(\mathcal{T})$ and $\mathbf{M}(\varphi)$ respectively.

• $\mathcal{T} \models \cdots, Th(\mathcal{T}), \mathbf{T}$: $\mathcal{T} \models \varphi$ and $\mathcal{T} \models \mathcal{T}_1$ are defined classically. A *theory* is a subset of $\mathbf{L}$ closed for deduction, and we denote by $\mathbf{T}$ the set $\{\mathcal{T} \subseteq \mathbf{L}\ /\ \mathcal{T} = Th(\mathcal{T})\}$ of the theories of $\mathbf{L}$.

• $\top, \bot,\ \sqcup$: Two logical constants $\top$ and $\bot$ denote respectively the true and the false formulas. If $\mathcal{T}_1, \mathcal{T}_2$ are subsets of $\mathbf{L}$, and φ is a formula in $\mathbf{L}$, we write $\mathcal{T}_1 \sqcup \mathcal{T}_2$ for $Th(\mathcal{T}_1 \cup \mathcal{T}_2)$ and $\mathcal{T} \sqcup \varphi$ for $Th(\mathcal{T} \cup \{\varphi\})$. Thus $\mathbf{M}(\mathcal{T}_1 \sqcup \mathcal{T}_2) = \mathbf{M}(\mathcal{T}_1 \cup \mathcal{T}_2) = \mathbf{M}(\mathcal{T}_1) \cap \mathbf{M}(\mathcal{T}_2)$.

• $\mathbf{C}, Th(\mu), Th(\mathbf{M}_1)$: A theory $\mathcal{C} \in \mathbf{T}$ is *complete* if $\forall \varphi \in \mathbf{L}$, $\varphi \in \mathcal{C}$ iff $\neg\varphi \notin \mathcal{C}$. We denote by $\mathbf{C}$ the set of all the complete theories of $\mathbf{L}$. $Th(\mu)$ denotes the set of the formulas satisfied by μ. For any subset $\mathbf{M}_1$ of $\mathbf{M}$, $Th(\mathbf{M}_1) = \{\varphi \in \mathbf{L}\ /\ \mu \models \varphi$ for any $\mu \in \mathbf{M}_1\} = \bigcap_{\mu \in \mathbf{M}_1} Th(\mu)$. This ambiguous use of Th and of $\models$ (applied to sets of formulas or to interpretations) is usual. For any $\mathcal{T} \in \mathbf{T}$, we get $\mathcal{T} = \bigcap_{\mathcal{C} \in \mathbf{C},\ \mathcal{C} \models \mathcal{T}} \mathcal{C}$. $\mathbf{C}$ can be assimilated to $\mathbf{M}$: For any $\mu \in \mathbf{M}$ we have $Th(\mu) \in \mathbf{C}$ and for any $\mathcal{C} \in \mathbf{C}$, $\mathbf{M}(\mathcal{C})$ is a singleton $\{\mu\} \subseteq \mathbf{M}$.

• TC: TC denotes the topological closure: $TC(\mathbf{M}_1) = \mathbf{M}(Th(\mathbf{M}_1))$ for $\mathbf{M}_1 \subseteq \mathbf{M}$. If $V(\mathbf{L})$ is finite, any subset of $\mathbf{M}$ is trivially closed and open.

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3 The two simplest preferential entailments

As different kinds of "preferential entailment" exist (see e.g. [3] about this issue) we will make the definitions precise. For didactical reasons, we begin by the simplest definition. Any "preferential entailment" is a particular case of an inference operation that we call a pre-circumscription:

Definition 3.1 A *pre-circumscription* f (in **L**) is an extensive (i.e., $f(\mathcal{T}) \supseteq \mathcal{T}$ for any $\mathcal{T}$) mapping from **T** to **T**. For any subset $\mathcal{T}$ of **L**, we use the abbreviation $f(\mathcal{T}) = f(Th(\mathcal{T}))$, assimilating a pre-circumscription to a particular extensive mapping from $\mathcal{P}(\mathbf{L})$ to itself[2]. We write $f(\varphi)$ for $f(\{\varphi\}) = f(Th(\varphi))$. □

Definitions 3.2 1. A *preference relation* in **L** is a binary relation $\prec$ over **M**. $\mathbf{M}_\prec(\mathcal{T})$ denotes the set of the models of $\mathcal{T}$ minimal for $\prec$: $\mathbf{M}_\prec(\mathcal{T}) = \{\mu \in \mathbf{M}(\mathcal{T})$ / for no $\nu \in \mathbf{M}(\mathcal{T})$ we have $\nu \prec \mu\}$.

2. The (ordinary) *preferential entailment* $f = f_\prec$ is the pre-circumscription in **L** defined by

$$f_\prec(\mathcal{T}) = Th(\mathbf{M}_\prec(\mathcal{T})). \quad \square$$

A well known kind of preferential entailment is circumscription [7, 11]. Definition 3.2 is the classical definition of preferential entailments originating in [13], applied to the propositional case. In the predicate calculus, any complete theory has as many models as we want which makes the notion of preferential entailment more powerful. We can simulate in the propositional case the main aspects of the preferential entailments in the predicate calculus case, thanks to a notion introduced in [5, Definition 5.6]:

Definitions 3.3 1. **S** is some set of *copies of elements of* **M**, called *states*: there exists a mapping l from **S** to **M**, and for any $\mu \in \mathbf{M}$ we call the set $l^{-1}(\mu) = \{\mu_1, \mu_2, \cdots\}$ the set (possibly empty) of the *copies of the interpretation* μ in **S**. $\mathbf{S}(\mathcal{T})$ is the subset of **S** defined by $\mathbf{S}(\mathcal{T}) = l^{-1}(\mathbf{M}(\mathcal{T}))$.

2. A *multi preference relation* in **L** is a binary relation $\prec_m$ over such a set **S**. For any $\mathcal{T} \in \mathbf{T}$, we define the sets $\mathbf{S}_{\prec_m}(\mathcal{T}) = \{\mu_i \in \mathbf{S}(\mathcal{T})$/ for no $\nu_j \in \mathbf{S}(\mathcal{T})$, we have $\nu_j \prec_m \mu_i\}$ and $\mathbf{M}_{\prec_m}(\mathcal{T}) = l(\mathbf{S}_{\prec_m}(\mathcal{T}))$.
3. A *multi preferential entailment* is a pre-circumscription defined by:

$$f_{\prec_m}(\mathcal{T}) = Th(\mathbf{M}_{\prec_m}(\mathcal{T})) \quad \square$$

Remarks 3.4 1. Any preferential entailment is a multi-preferential entailment: choose $\mathbf{S} = \mathbf{M}$ and $l = identity$.

2. $f_{\prec_m}(\mathcal{T}) \models \varphi$ iff $\mu \models \varphi$ for any $\mu \in \mathbf{M}_{\prec_m}(\mathcal{T})$.

3. $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = TC(\mathbf{M}_{\prec_m}(\mathcal{T}))$, $f_{\prec_m}(\mathcal{T}) = \bigcap_{\mu \in \mathbf{M}_{\prec_m}(\mathcal{T})} Th(\mu)$.

3′. Thus, if $V(\mathbf{L})$ is finite, $\mathbf{M}(f_{\prec_m}(\mathcal{T})) = \mathbf{M}_{\prec_m}(\mathcal{T})$. □

4 A menagerie of properties

Definitions 4.1 Here are various properties a pre-circumscription may possess. φ is a formula in **L**, $\mathcal{T}, \mathcal{T}''$ are sets of formulas in **L**, while $\mathcal{T}_1, \mathcal{T}_2$ are in **T**:

Idempotence: $f(f(\mathcal{T})) = f(\mathcal{T})$. **(Idem)**

[2] Thus, for a reader familiar with the terminology used in [5], a pre-circumscription is an *inference operation* satisfying the full (or theory) versions of "*reflexivity*", "*left logical equivalence LLE*", "*right weakening RW*" and "*AND*".

Reverse monotony: $f(\mathcal{T} \cup \mathcal{T}'') \subseteq f(\mathcal{T}) \sqcup \mathcal{T}''$. **(RM)**

Case reasoning: $f(\mathcal{T}_1) \cap f(\mathcal{T}_2) \subseteq f(\mathcal{T}_1 \cap \mathcal{T}_2)$. **(CR)**

Disjunctive coherence: $f(\mathcal{T}_1 \cap \mathcal{T}_2) \subseteq f(\mathcal{T}_1) \sqcup f(\mathcal{T}_2)$. **(DC)**

Monotony: $f(\mathcal{T}) \subseteq f(\mathcal{T} \cup \mathcal{T}'')$. **(MON)**

Cumulative transitivity: If $\mathcal{T}'' \subseteq f(\mathcal{T})$, $f(\mathcal{T} \cup \mathcal{T}'') \subseteq f(\mathcal{T})$. **(CT)**

Cumulative monotony: If $\mathcal{T}'' \subseteq f(\mathcal{T})$, $f(\mathcal{T}) \subseteq f(\mathcal{T} \cup \mathcal{T}'')$. **(CM)**

Cumulativity: If $\mathcal{T}'' \subseteq f(\mathcal{T})$ then $f(\mathcal{T}) = f(\mathcal{T} \cup \mathcal{T}'')$. **(CUMU)**

(LOOP$_n$): If $\mathcal{T}_2 \subseteq f(\mathcal{T}_1), \cdots, \mathcal{T}_n \subseteq f(\mathcal{T}_{n-1}), \mathcal{T}_1 \subseteq f(\mathcal{T}_n)$, then $f(\mathcal{T}_1) = f(\mathcal{T}_n)$ **(LOOP$_n$)**

(LOOP): For any integer $n \geq 2$, f satisfies (LOOP$_n$). **(LOOP)** □

Definitions 4.2 We also need some weaker versions.

1. Formula versions

(RM1) $f(\mathcal{T} \sqcup \varphi) \subseteq f(\mathcal{T}) \sqcup \varphi$.

(CR1) $f(\mathcal{T} \sqcup \varphi) \cap f(\mathcal{T} \sqcup \psi) \subseteq f(\mathcal{T} \sqcup \varphi \vee \psi)$.

2. Formula-only versions

(RM0) $f(\psi \wedge \varphi) \subseteq f(\psi) \sqcup \varphi$. **(CR0)** $f(\varphi) \cap f(\psi) \subseteq f(\varphi \vee \psi)$.

(DC0) $f(\varphi \vee \psi) \subseteq f(\varphi) \sqcup f(\psi)$.

(CUMU0) If $\varphi \in f(\psi)$ then $f(\psi) = f(\varphi \wedge \psi)$. □

Each of these properties has an immediate interpretation in terms of reasoning. For instance, (CR1) means that if we know that generally birds fly, formalized as $Fly_i \in f(\mathcal{T} \sqcup Bird_i)$, and that generally bats fly, formalized as $Fly_i \in f(\mathcal{T} \sqcup Bat_i)$, then if all we know a priori about individual i is that it is a bird or a bat, we conclude that i flies ($Fly_i \in f(\mathcal{T} \sqcup Bird_i \vee Bat_i)$).

Property 4.3 (folklore, and immediate) *For pre-circumscriptions:*

1. *(RM1) and (CR1) are equivalent, as are (RM0) and (CR0).*
2. *(RM) implies (CT) and (CR), (CT) implies (Idem).*
3. *Any full version implies its formula version, any formula version implies its formula-only version.*
4. *(LOOP$_2$) is equivalent to (CUMU), (LOOP$_{n+1}$) is strictly stronger than (LOOP$_n$) for any $n \geq 2$* (see example 5.3 below which can easily be generalized).
5. *(CR) and (CUMU) imply (LOOP). Thus, a multi preferential entailment satisfies (LOOP) iff it satisfies (CUMU) iff it satisfies (CM)* (see property 5.14 below). □

Here are two important properties that a (multi) preference relation may posses.

Definition 4.4 In 1, $\prec$ denotes either a preference relation (denoted by $\prec$ in definition 3.2) or a multi preference relation (denoted by $\prec_m$ in definition 3.3).

1. A (multi) preference relation $\prec$ *satisfies the closure property* (or *is (cl)*), if for any $\mathcal{T} \in \mathbf{T}$, $\mathbf{M}_\prec(\mathcal{T})$ is a closed set: $\mathbf{M}_\prec(\mathcal{T}) = \mathbf{M}(f_\prec(\mathcal{T}))$.
2. A multi preference relation $\prec_m$ is *safely founded (sf)* if, for any $\mu_i \in \mathbf{S}(\mathcal{T}) - \mathbf{S}_{\prec_m}(\mathcal{T})$, there exists $\nu_j \in \mathbf{S}_{\prec_m}(\mathcal{T})$ such that $\nu_j \prec_m \mu_i$.
   A preference relation $\prec$ is *(sf)* if, for any $\mu \in \mathbf{M}(\mathcal{T}) - \mathbf{M}_\prec(\mathcal{T})$, there exists $\nu \in \mathbf{M}_\prec(\mathcal{T})$ such that $\nu \prec \mu$. □

(cl) is *definability preserving* in [12], close to *fullness property* in [4], and *faithful* in [6]. (Multi) preferential entailments in which the relation is (cl) are simpler because we get $\mathbf{M}(f_\prec(\mathcal{T})) = \mathbf{M}_\prec(\mathcal{T})$. If $V(\mathbf{L})$ is finite, (cl) is trivially satisfied.

(sf) appears for propositional multi preference relations as *smooth* in [5] and *stoppered* in [6].

5 General preferential entailment

5.1 Definition and first properties

Definitions 5.1 1. For any $\mathcal{T} \subseteq \mathbf{L}$ we define the subset of $\mathbf{T}$: $\mathbf{W}(\mathcal{T}) = \{\mathcal{T}_1 \in \mathbf{T} \ / \ \mathcal{T} \subseteq \mathcal{T}_1\}$. We write $\mathbf{W}(\varphi)$ for $\mathbf{W}(\{\varphi\})$.

2. $\mathbf{S}$ is some set of copies of elements of $\mathbf{T}$, also called *states*: there exists a mapping l from $\mathbf{S}$ to $\mathbf{T}$. As usual, we define $l(\mathbf{S}) = \{l(s)\}_{s \in \mathbf{S}} = \{\mathcal{T} \in \mathbf{T} \ / \ l^{-1}(\mathcal{T}) \neq \emptyset\}$. For any $\mathcal{T} \subseteq \mathbf{L}$, $\mathbf{S}(\mathcal{T})$ is the subset of $\mathbf{S}$ defined by $\mathbf{S}(\mathcal{T}) = l^{-1}(\mathbf{W}(\mathcal{T}))$.
3. A *general preference relation* $\prec_g$ is a binary relation over $\mathbf{S}$. For any $\mathcal{T} \in \mathbf{T}$, we define the sets $\mathbf{S}_{\prec_g}(\mathcal{T}) = \{s \in \mathbf{S}(\mathcal{T}) \ / \ s_1 \prec_g s$ for no $s_1 \in \mathbf{S}(\mathcal{T})\}$, and $\mathbf{W}_{\prec_g}(\mathcal{T}) = l(\mathbf{S}_{\prec_g}(\mathcal{T}))$.
4. The *general preferential entailment* $f_{\prec_g}$ is defined by $f_{\prec_g}(\mathcal{T}) = \bigcap_{\mathcal{T}_1 \in \mathbf{W}_{\prec_g}(\mathcal{T})} \mathcal{T}_1$ for any $\mathcal{T} \subseteq \mathbf{L}$.
5. A general preference relation $\prec_g$ is *(sf)*, if for any $s \in \mathbf{S}(\mathcal{T}) - \mathbf{S}_{\prec_g}(\mathcal{T})$, there exists $s_1 \in \mathbf{S}_{\prec_g}(\mathcal{T})$ such that $s_1 \prec_g s$. □

This notion appears in [5, Definitions 3.11 and 3.13] and [4, Definition 4.26] (except that there $\prec_g$ must be (sf)), and in [3, Definitions 3.1 and 3.2] (without restriction). In [5, 4], l_{klm}, is a mapping from $\mathbf{S}$ to $\mathcal{P}(\mathbf{M})$ (and the relation $\prec_{klm}$ is among sets of interpretations instead of theories), but, as noted in [3], this makes no difference: it suffices to define $l(s) = Th(l_{klm}(s))$. Our definition is similar to the formulation in remark 3.4-3 for (multi) preferential entailments. Here is an alternative formulation for definition 5.1-4:

Remark 5.2 For any $\varphi \in \mathbf{L}$, $\varphi \in f_{\prec_g}(\mathcal{T})$ iff $\mathbf{W}_{\prec_g}(\mathcal{T}) \subseteq \mathbf{W}(\varphi)$. □

The following characterization result is known:

Theorem 5.3 [5, Theorem 3.25] [4, Theorems 4.30 and 4.35] *A pre-circumscription f is a general preferential entailment $f_{\prec_g}$ defined by a relation which is (sf) iff it satisfies (CUMU).* □

We will drop the (sf) condition, studying "pure" general preferential entailments, and we will show that a similar result exists.

Any multi-preferential entailment is a general preferential entailment, corresponding to a mapping l from $\mathbf{S}$ to $\mathbf{C} \subseteq \mathbf{T}$ (thus $l(\mathbf{S}) \subseteq \mathbf{C}$). We will prove (see definition 5.7 and theorem 5.10 below) that definition 5.1 could be simplified: we could take $\mathbf{S} = \mathbf{T}$ and $l = identity$. However, the connexion with multi preferential entailments is not so immediate with this simplified form.

Lemma 5.4 For any subsets $\mathcal{T}_1, \mathcal{T}_2$ of $\mathbf{L}$ we have:
1. $\mathcal{T}_2 \subseteq f_{\prec_g}(\mathcal{T}_1)$ iff $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}(\mathcal{T}_2)$.
2. If $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}(\mathcal{T}_2)$ then $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}_{\prec_g}(\mathcal{T}_1 \cup \mathcal{T}_2)$.

Proof: 1. If $\mathcal{T}_2 \subseteq f_{\prec_g}(\mathcal{T}_1)$, then for any $\varphi \in \mathcal{T}_2$ we have $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}(\varphi)$, thus $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \bigcap_{\varphi \in \mathcal{T}_2} \mathbf{W}(\varphi) = \mathbf{W}(\mathcal{T}_2)$. If $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}(\mathcal{T}_2)$, then for any $\varphi \in \mathcal{T}_2$ we get $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}(\varphi)$, i.e., $\varphi \in f_{\prec_g}(\mathcal{T}_1)$. (This result extends remark 5.2 to theories which are not finitely axiomatizable.)

2. $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}(\mathcal{T}_2)$ and $\mathcal{T} \in \mathbf{W}_{\prec_g}(\mathcal{T}_1)$: Then there exists $s \in \mathbf{S}_{\prec_g}(\mathcal{T}_1)$ such that $l(s) = \mathcal{T} \in \mathbf{W}(\mathcal{T}_2) \cap \mathbf{W}(\mathcal{T}_1) = \mathbf{W}(\mathcal{T}_1 \cup \mathcal{T}_2)$. If $s' \in \mathbf{S}(\mathcal{T}_1 \cup \mathcal{T}_2) = \mathbf{S}(\mathcal{T}_1) \cap \mathbf{S}(\mathcal{T}_2)$ we get $s' \not\prec_g s$ from $s \in \mathbf{S}_{\prec_g}(\mathcal{T}_1)$. Thus, $\mathcal{T} \in \mathbf{W}_{\prec_g}(\mathcal{T}_1 \cup \mathcal{T}_2)$. □

Property 5.5 *Any general preferential entailment $f = f_{\prec_g}$ is a pre-circumscription satisfying (CT).*

Remind that (CT) implies (Idem) (property 4.3-2).
Proof: Let us suppose $\mathcal{T}_2 \subseteq f(\mathcal{T}_1)$, i.e. (lemma 5.4-1), $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}(\mathcal{T}_2)$. We suppose also $\varphi \in f(\mathcal{T}_1 \cup \mathcal{T}_2)$, i.e. from remark 5.2, $\mathbf{W}_{\prec_g}(\mathcal{T}_1 \cup \mathcal{T}_2) \subseteq \mathbf{W}(\varphi)$. Then from lemma 5.4-2 we get $\mathbf{W}_{\prec_g}(\mathcal{T}_1) \subseteq \mathbf{W}(\varphi)$, i.e., $\varphi \in f(\mathcal{T}_1)$: this establishes $f(\mathcal{T}_1 \cup \mathcal{T}_2) \subseteq f(\mathcal{T}_1)$. Thus, f satisfies (CT). □

5.2 A characterization result

Any general preferential entailment satisfies (CT). We will prove the converse and provide a simple form for general preferential entailments, showing that this notion is "overly defined", and that it looks more "*cumbersome*" [5] than it is.

Definition 5.6 Let us call *simplified* any general preference relation $\prec_g$ where $\mathbf{S} = \mathbf{W}$ and l is the identity. In this case, the set $\mathbf{S}$ of states is useless. □

Notice that it is immediate to show that if $V(\mathbf{L})$ is finite, then any simplified general preference relation which is transitive and irreflexive is (sf) (to be compared with [5, Part 4.1]).

Definition 5.7 f being a pre-circumscription in $\mathbf{L}$, we define three relations on the set $\mathbf{T}$, as follows:
$\mathcal{T}_1 \prec_1 \mathcal{T}_2$ if $f(\mathcal{T}_2) = \mathcal{T}_1$ and $\mathcal{T}_1 \neq \mathcal{T}_2$.
$\mathcal{T}_1 \prec_2 \mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ and $f(\mathcal{T}_1) \not\subseteq \mathcal{T}_2$.
$\mathcal{T}_1 \prec_f \mathcal{T}_2$ if $\mathcal{T}_1 \prec_1 \mathcal{T}_2$ or $\mathcal{T}_1 \prec_2 \mathcal{T}_2$. □

Remarks 5.8 1. $\prec_1$, $\prec_2$ and $\prec_f$ are irreflexive. If f satisfies (Idem), $\prec_1$ is trivially transitive: we have never $\mathcal{T}_1 \prec_1 \mathcal{T}_2 \prec_1 \mathcal{T}_3$.
$\prec_2$ and $\prec_f$ are not necessarily transitive.
2. Let us suppose that we have $\mathcal{T}_0 \prec_f \mathcal{T}$ for some $\mathcal{T}_0, \mathcal{T}_1$ in $\mathbf{T}$:
- We have $\mathcal{T}_0 \prec_1 \mathcal{T}$ iff $\mathcal{T} \subseteq \mathcal{T}_0$ (in this case we have $\mathcal{T} \subset f(\mathcal{T}) = \mathcal{T}_0$).
- We have $\mathcal{T}_0 \prec_2 \mathcal{T}$ iff $\mathcal{T} \not\subseteq \mathcal{T}_0$. □

Lemma 5.9 If there exists some $\mathcal{T}' \in \mathbf{T}$ such that $\mathcal{T}' \in \mathbf{W}_{\prec_g}(\mathcal{T}) \subseteq \mathbf{W}(\mathcal{T}')$, then $f_{\prec_g}(\mathcal{T}) = \mathcal{T}'$.

Proof: From $\mathbf{W}_{\prec_g}(\mathcal{T}) \subseteq \mathbf{W}(\mathcal{T}')$ we get $\bigcap_{\mathcal{T}_1 \in \mathbf{W}_{\prec_g}(\mathcal{T})} \mathcal{T}_1 \supseteq \mathcal{T}'$. From $\mathcal{T}' \in \mathbf{W}_{\prec_g}(\mathcal{T})$ we get then $\mathcal{T}' = \bigcap_{\mathcal{T}_1 \in \mathbf{W}_{\prec_g}(\mathcal{T})} \mathcal{T}_1$. □

Theorem 5.10 *If f is a pre-circumscription satisfying (CT), then $f = f_{\prec_f}$.*

Proof: 1) $f(\mathcal{T}) \in \mathbf{W}_{\prec_f}(\mathcal{T})$: Let us suppose $f(\mathcal{T}) = \mathcal{T}$. For no $\mathcal{T}_0$ we have $\mathcal{T}_0 \prec_1 \mathcal{T}$. As for any $\mathcal{T}_0 \in \mathbf{W}(\mathcal{T})$ we have $\mathcal{T} \subseteq \mathcal{T}_0$, for no $\mathcal{T}_0 \in \mathbf{W}(\mathcal{T})$ we have $\mathcal{T}_0 \prec_2 \mathcal{T}$. Thus we get $f(\mathcal{T}) = \mathcal{T} \in \mathbf{W}_{\prec_f}(\mathcal{T})$.

Let us suppose now $f(\mathcal{T}) \neq \mathcal{T}$. Defining $\mathcal{T}_0 = f(\mathcal{T})$, we get $f(\mathcal{T}_0) = \mathcal{T}_0$ from (Idem), a consequence of (CT) (property 4.3-2). Thus, for no $\mathcal{T}' \in \mathbf{T}$ we have $\mathcal{T}' \prec_1 \mathcal{T}_0$. Let us suppose that there exists $\mathcal{T}' \in \mathbf{W}(\mathcal{T})$ such that $\mathcal{T}' \prec_2 \mathcal{T}_0$. Then $\mathcal{T} \subseteq \mathcal{T}'$ from $\mathcal{T}' \in \mathbf{W}(\mathcal{T})$, and $\mathcal{T}' \subset \mathcal{T}_0$ and $f(\mathcal{T}') \not\subseteq \mathcal{T}_0$ from $\mathcal{T}' \prec_2 \mathcal{T}_0$. Thus $\mathcal{T}' \subset f(\mathcal{T})$, thus, from (CT), $f(\mathcal{T} \cup \mathcal{T}') \subseteq f(\mathcal{T})$ with $\mathcal{T} \cup \mathcal{T}' = \mathcal{T}'$ thus $f(\mathcal{T}') \subseteq f(\mathcal{T})$, i.e., $f(\mathcal{T}') \subseteq \mathcal{T}_0$, a contradiction.

2) $\mathbf{W}_{\prec_f}(\mathcal{T}) \subseteq \mathbf{W}(f(\mathcal{T}))$: Let us suppose $\mathcal{T}_0 \in \mathbf{W}_{\prec_f}(\mathcal{T})$. Then we get $\mathcal{T}_0 \in \mathbf{W}(\mathcal{T})$, i.e. $\mathcal{T} \subseteq \mathcal{T}_0$. If $\mathcal{T}_0 \neq \mathcal{T}$ we get $f(\mathcal{T}) \subseteq \mathcal{T}_0$, as otherwise we would get $\mathcal{T} \prec_2 \mathcal{T}_0$, a contradiction with $\mathcal{T}_0 \in \mathbf{W}_{\prec_f}(\mathcal{T})$. If $\mathcal{T}_0 = \mathcal{T}$ then, if $\mathcal{T} \neq f(\mathcal{T})$, we get $f(\mathcal{T}) \prec_1 \mathcal{T}$, a contradiction with $\mathcal{T} = \mathcal{T}_0 \in \mathbf{W}_{\prec_f}(\mathcal{T})$. Thus we get $\mathcal{T} = f(\mathcal{T}) \subseteq \mathcal{T}_0 (= \mathcal{T})$. In any case, we get $\mathcal{T}_0 \in \mathbf{W}(f(\mathcal{T}))$.

3) From 1, 2 and lemma 5.9, we get $f(\mathcal{T}) = f_{\prec_f}(\mathcal{T})$. □

Here is an immediate consequence of property 5.5 and theorem 5.10:

Corollary 5.11 *For any general preferential entailment $f_{\prec_g}$, there exists a simplified general preference relation $\prec_{gs}$ such that $f_{\prec_g} = f_{\prec_{gs}}$.* □

Thus, the definition of general preferential entailments could have been simplified. This "elimination of the states" in definition 5.1, was partially known, but only if the general preferential entailment satisfies (CUMU) [1, 2] (more precisely satisfies the formula-only version (CUMU0)), or another very strong condition (called "rational monotony") [1]. The present method is the only one applying to any general preferential entailment. Moreover it is much simpler than the methods given in [1] and even in [2], and it allows to give directly the result of f from the simplified relation considered here (property 5.13 below).

Here is another immediate consequence of property 5.5 and theorem 5.10, to be compared with property 5.14 (given below).

Theorem 5.12 *A pre-circumscription satisfies (CT) iff it is a general preferential entailment.*

Property 5.13 *Let f be a general preferential entailment, and $\prec_f$ be the relation introduced in definition 5.7, then we have, for any $\mathcal{T} \in \mathbf{T}$, $f(\mathcal{T}) = \mathcal{T}_0$ iff*
$(\mathcal{T}_0 \prec_1 \mathcal{T})$ or $(\mathcal{T}_0 = \mathcal{T}$ and, for no $\mathcal{T}' \in \mathbf{T}$, we have $\mathcal{T}' \prec_1 \mathcal{T})$.

A reminder from remark 5.8-2: *It is easy to get $\prec_1$ when we know $\prec_f$. Indeed, $\mathcal{T}_0 \prec_1 \mathcal{T}$ iff $\mathcal{T}_0 \prec_f \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{T}_0$. Thus it is immediate to get $f(\mathcal{T})$ when we know $\prec_f$.*

Proof: This is an immediate consequence of remark 5.8-2. Notice that $\prec_f$ is such that for any $\mathcal{T} \in \mathbf{T}$, there exists one and only one $\mathcal{T}_0 \in \mathbf{T}$ satisfying the conditions given here. □

So, each time f is a general preferential entailment, we have described a general preference relation associated to f, and even a simplified relation, which is such that it is immediate to get the value of $f(\mathcal{T})$ directly from $\mathcal{T}$ and from this relation, without making the computations of definition 5.1 or remark 5.2.

Thus, we have exhibited an easy passage, in the two directions, between the notion of pre-circumscription satisfying (CT) and the notion of general preferential entailment. Starting from a pre-circumscription satisfying (CT), we use definition 5.7 and we get a (simplified) general preference relation. Starting from a general preferential entailment defined by the simplified general preference relation introduced in definition 5.7, we have shown how the pre-circumscription can be obtained directly from the relation.

5.3 About the "simplest" preferential entailments

We have no room for detailing the characterizations of the simplest preferential entailments (see [9], which translates to the propositional case the results for the predicate case, from 1994, of the still unpublished [10]). They are more complicated and less friendly than for general preferential entailments. We only list the main results known in the literature here. "Friendly" characterization results are known for multi preferential entailments whose relation is (cl) [12, Theorem 3.1] thus for finite multi preferential entailments (the condition being (RM) alone), and for multi preferential entailments whose relation is (cl)+(sf) (the condition being (RM)+(CM)) ([5, Theorem 5.18], [4, Theorem 7.27]). It is likely that no characterization of pure multi preferential entailment exists by properties as natural as (RM) or (DC). Multi preferential entailment may falsify (RM) [12], however they satisfy two important consequences of (RM):

Property 5.14 (cf e.g. [6]) *Any multi preferential entailment satisfies (CT)* (cf property 5.5) *and (CR), thus (Idem) and (RM1).* □

Thus, the main difference between multi preferential entailments and general preferential entailments is that the former satisfies (CR) and (CT) while the latter satisfies only (CT). If $V(\mathbf{L})$ is infinite, a pre-circumscription satisfying (CT) and (CR) is not necessarily a multi preferential entailment [10]. Thus, we need another property of pre-circumscriptions, called (CP):

Definitions 5.15 [8, 9] 1. f satisfies the *property of common points*, or (CP) if, for any $\mathcal{T} \cup \{\varphi\} \subseteq \mathbf{L}$ with $f(\mathcal{T}) \not\models \neg\varphi$, there exists $\mu \in \mathbf{M}(f(\mathcal{T}) \sqcup \varphi)$ with $\mu \in \bigcap_{\mathcal{T}'' \in \mathbf{W}(\mathcal{T}),\ \mu \models \mathcal{T}''} \mathbf{M}(f(\mathcal{T}''))$. **(CP)**

2. $Cp_f(\mathcal{T}) = \{\mu \in \mathbf{M}(f(\mathcal{T})) / \mu \in \mathbf{M}(\bigsqcup_{\mathcal{T}'' \in \mathbf{W}(\mathcal{T}), \mu \models \mathcal{T}''} f(\mathcal{T}''))\}$.

Theorem 5.16 ([8, Remark 3.9-1], [9, Property 6.2 and Theorem 6.11]) *A pre-circumscription f satisfies* (CP) *iff $\mathbf{M}(f(\mathcal{T})) = TC(Cp_f(\mathcal{T}))$ for any $\mathcal{T}$, iff it is a multi-preferential entailment.* □

We turn now to preferential entailments:

Definition 5.17 1. [9, Definition 7.2] $\mathbf{M}_f(\mathcal{T}) = \{\mu \in \mathbf{M}(\mathcal{T})\ /\ \forall \nu \in \mathbf{M}(\mathcal{T}),\ \mu \in \mathbf{M}(f(Th(\{\mu, \nu\})))\}$.

2. (e.g. [8, Proposition 3.8-2]) A pre-circumscription f *satisfies* (DCC) if, for any $\mathcal{T} \subseteq \mathbf{L}$, $\mathbf{M}_f(\mathcal{T})) \subseteq \mathbf{M}(f(\mathcal{T}))$. **(DCC)**
3. [6, 8, 9] f satisfies (P$'$) if, for any $\varphi \in \mathbf{L}$ and any $\mathcal{T} \subseteq \mathbf{L}$ such that $f(\mathcal{T}) \not\models \neg\varphi$, we have $\mathbf{M}(f(\mathcal{T}) \sqcup \varphi) \cap \mathbf{M}_f(\mathcal{T}) \neq \emptyset$. **(P$'$)** □.

Theorem 5.18 ([8, Proposition 3.8-2], [9, Theorems 7.13 and 7.17]) *A pre-circumscription f is a preferential entailment iff f satisfies (DCC) and (CP), iff f satisfies (DCC) and (P$'$).* □

Again, it is only in the finite case that an easy characterization result exists, originating in [11]: If $V(\mathbf{L})$ is finite, then a pre-circumscription f is a preferential entailment iff it satisfies (RM) and (DC). In the infinite case, a preferential entailment may falsify (DC). Any preferential entailment satisfying (RM) (i.e. whose relation is (cl)) satisfies (DC), however a pre-circumscription may satisfy (RM) and (DC) without being a preferential entailment [10].

It may seem paradoxical that the apparently most intricate version of "preferential entailment" possesses in fact, by far, the simplest characterization in terms of logical properties.

5.4 Three examples of finite general preferential entailment

As we are in the finite case, we may assimilate as usual **T** to **L** and f to a mapping from **L** to **L** (writing $Th(f(\varphi))$ when necessary).

Firstly, we give the simplest example of general preferential entailment which is not a multi preferential entailment, i.e. in the finite case, of a pre-circumscription satisfying (CT) and falsifying (CR).

Example 5.1 $V(\mathbf{L}) = \{P\}$,
$f(\varphi) = \bot$ if $\varphi \in \{\bot, P, \neg P\}$ and $f(\top) = \top$.

f falsifies (CR): $f(P \vee \neg P) \not\models f(P) \vee f(\neg P)$. f satisfies (CT): it satisfies (Idem) and, if $\mathcal{T}_1 \subset Th(f(\varphi))$ then $f(\varphi) = \bot$.

Definition 5.7 gives $\bot \prec_1 P,\ \bot \prec_1 \neg P$, $\prec_f = \prec_1$ ($\prec_2$ has an empty graph here). It is easy to check that $f = f_{\prec_f}$.

Notice that f satisfies also (LOOP) here, thus (CUMU): Let us suppose that we have $f(\varphi_2) \models \varphi_1$, $f(\varphi_3) \models \varphi_1, \cdots, f(\varphi_n) \models$

φ_{n-1} and $f(\varphi_1) \models \varphi_n$. If $f(\varphi_i) = \varphi_i$, we get, using addition and subtraction modulo n, $f(\varphi_{i+1}) \models \varphi_i$, $f(\varphi_i) = \varphi_i \models \varphi_{i-1}$, thus $f(\varphi_{i+1}) \models \varphi_{i-1}$: if $n \geq 2$ we can thus suppress φ_i from the sequence. Now, as there are only two formulas $\varphi \in \mathbf{L}$ such that $f(\varphi) \neq \varphi$, the only possibility remaining is with a sequence of two elements. This establishes that we are in a case where f satisfies (LOOP) iff it satisfies (LOOP$_2$), i.e. (CUMU). As we have (CT) already, we prove (CM). If $Th(\psi) \subset Th(f(\varphi))$, we have $\varphi \neq \top$ thus $\varphi \wedge \psi \neq \top$ and $Th(f(\varphi)) \subseteq Th(f(\varphi \wedge \psi))$: f satisfies (CM). □

We give now the simplest general preferential entailment which falsifies (CM).

Example 5.2 $V(\mathbf{L}) = \{P\}$,
$f(\varphi) = \bot$ if $\varphi \in \{\bot, P, \top\}$ and $f(\neg P) = \neg P$.

It is easy to check that f satisfies (CR) or equivalently in the finite case (RM), thus f satisfies (CT). However, f falsifies (CM): $f(\top) \not\models \neg P$ while $f(\neg P \wedge \top) = f(\neg P) \not\models f(\top)$.

Definition 5.7 gives $\bot \prec_1 P$, $\bot \prec_1 \top$, $\top \prec_2 \neg P$ and $\varphi_1 \prec_f \varphi_2$ if $\varphi_1 \prec_1 \varphi_2$ or $\varphi_1 \prec_2 \varphi_2$. It is easy to check that $f = f_{\prec_f}$. Notice however that we get $f \neq f_{\overline{\prec_f}}$ where $\overline{\prec_f}$ denotes the transitive closure of $\prec_f$: indeed $f_{\overline{\prec_f}}(\neg P) = \bot$.

Notice that f satisfies also (DC) here, thus (see subsection 5.3), f is in fact a preferential entailment. And indeed, if we define the preference relation $\prec$ in $\mathbf{M} = \{\emptyset, \{P\}\}$ by $\{P\} \prec \{P\}$ and $\{P\} \prec \emptyset$, we get $f = f_\prec$.

Here is an example showing that (CUMU) does not imply (LOOP) for general preferential entailments, contrarily to what happens with multi preferential entailments.

Example 5.3 $V(\mathbf{L}) = \{A, B\}$. f is defined as follows:
$f(A \vee B) = A$, $f(\neg A \vee B) = B$, $f(A \vee \neg B) = A \Leftrightarrow B$, $f(\varphi) = \varphi$ for all the thirteen other formulas $\varphi \in \mathbf{L}$.

f satisfies (CUMU)=(LOOP$_2$): Let us suppose that we have $f(\varphi) \models \psi$ and $\psi \models \varphi$. Then, either $\varphi = \psi$ or we are in one of the three following cases: $(\varphi, \psi) \in \{(A, A \vee B),\ (B, \neg A \vee B),\ (A \Leftrightarrow B,\ A \vee \neg B)$. In each case we get $f(\varphi) = f(\psi)$.

f falsifies (LOOP$_3$): $f(A \vee B) \models A \vee \neg B$, $f(A \vee \neg B) \models \neg A \vee B$, $f(\neg A \vee B) \models A \vee B$, while $f(A \vee B) \neq f(\neg A \vee B)$.

As (CUMU) implies (CT), we know that f is a general preferential entailment and that we have $f = f_{\prec_f}$ where $\prec_f$ is as in definition 5.7. The graph of $\prec_f$ is the union of the following two graphs:
1. graph of $\prec_1$: $\{(A, A \vee B), (B, \neg A \vee B), (A \Leftrightarrow B, A \vee \neg B)\}$.
2. graph of $\prec_2$: $\{(A \vee B, B), (A \vee B, A \not\Leftrightarrow B), (A \vee B, \neg A \wedge B), (\neg A \vee B, A \Leftrightarrow B), (\neg A \vee B, \neg A), (\neg A \vee B, \neg A \wedge \neg B), (A \vee \neg B, A), (A \vee \neg B, \neg B), (A \vee \neg B, A \wedge \neg B)\}$.
As in example 5.2, $f \neq f_{\overline{\prec_f}}$: for instance $f_{\overline{\prec_f}}(A) = A \wedge B$. □

5.5 The first order case

Theorems 5.10 and 5.12 apply to predicate general preferential entailments: the states are useless here also and theorem 5.12 applies. If we define predicate general preference relations as relations among copies of classes of interpretations as done in [5] or among copies of theories as in definition 5.1 (in the propositional case), then we can obtain an equivalent (simplified) general preference relation defined in $\mathbf{T}$. As in the propositional case, but this may be found more important in this case, any (ordinary) predicate preferential entailment is a predicate general preferential entailment. Notice that predicate preferential entailments defined as in definition 3.2 (such as predicate circumscriptions) behave like propositional **multi** preferential entailments because now a complete theory has more than one model [9].

6 Conclusion

We have characterized the apparently most complicated kind of "preferential entailment" known in the literature, in terms of a very simple purely syntactical property. We have compared this result with the corresponding results for the "simplest" preferential entailments. We have also shown that an apparently yet more general version appearing in the literature is in fact overly general, as it can be reduced to a simpler version. Indeed, we do not need to use "states", which are copies of theories, we can define directly a simplified general preference relation directly among theories (or equivalently sets of interpretations). We have provided an easy way to define such a relation, each time it is possible, that is each time the property of cumulative transitivity is satisfied. It was already known that the states are useless for some particular cases of general preferential entailments. We have shown that there does not exist any situation in which this apparent additional flexibility is really significative. This simplifies the study of this notion.

A lot of work still remains. A by-product of our results is that any predicate preferential entailment (such as any classical circumscription) can be described as a general preferential entailment, that is by a relation among theories. It remains to investigate this point and to give direct translations from the original relation between interpretations to a relation between theories. Various possible relations should be studied, in order to find an interesting one which could possibly help our understanding of the notion studied, or the automatic computation. Also, we could try to find some more precise results: when we have a general preferential entailment, are there "natural" conditions, but not too strong, which make that it is in fact a simpler kind of preferential entailment? Again, the impact of such a study could be on the knowledge representation side, helping to get a better understanding of complex notions, and on the computational side.

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