Polynomial algorithms for clearing multi-unit single-item and multi-unit combinatorial reverse auctions

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Abstract. This paper develops new algorithms for clearing multi-unit single-item and multi-unit combinatorial reverse auctions. Specifically, we consider settings where bids are submitted in the form of a supply function and the auctions have sub-additive pricing with free disposal. Our algorithms are based on a greedy strategy and we show that they are of polynomial complexity. Furthermore, we show that the solutions they generate are within a finite bound of the optimal.

1 INTRODUCTION

Online auctions are a key enabling component of e-commerce since they are an efficient method of allocating goods/services in dynamic situations to the agents who value them most highly [8]. Traditionally, the most common forms of online auction are the simple, single-sided auctions in which a single item is traded (examples of such protocols are English, Dutch, first price sealed-bid and Vickrey auctions). However as more trading takes place in such environments, we believe their inherent limitations will become more apparent. This will, in turn, increase the demand for more sophisticated marketplaces in which multiple units of multiple (potentially inter-related) items are traded simultaneously. Such auctions are called *multi-unit combinatorial auctions* [8]. In this type of auction, bidders may bid for arbitrary combinations of items. For example, a single bid may be for q units of item 1 and $2 * q$ units of item 2 at price $40 * q$ if $q < 20$, and at price $30 * q$ if $q \geq 20$. This degree of flexibility in expressing requirements, we believe, will be especially useful in business-to-business e-commerce where there is often a need to trade multiple, inter-related goods or services on a massive scale.

While multi-unit combinatorial auctions have many potential benefits from an economic perspective [1], their main disadvantages stem from the lack of computationally tractable clearing (winner determination) algorithms for determining the prices, quantities and trading partners as a function of the bids made. Without such algorithms, multi-unit combinatorial auctions are simply not practicable. To overcome this problem, there has been considerable recent work in this area (e.g. [1], [2], [3], [4], [6], [7]). However, almost all of this work has considered bids to be *atomic propositions* that are either accepted in their entirety or rejected. This view can limit the potential profit available to the auctioneer. For example, consider the case where there are only two bids: x_1 units of one good at price p_1 and x_2 units at price p_2, and the quantity the auctioneer wants to trade is less than $x_1 + x_2$ units . In this case, the auctioneer has no choice other than selecting one or other of the two bids. This may prevent the auctioneer from maximising its payoff. For example, the auctioneer may find it more beneficial to accept both bids partially; that is, trade y_1 ($y_1 < x_1$) units with bidder 1 at price $\frac{y_1}{x_1} \cdot p_1$ and trade y_2 ($y_2 < x_2$) units with bidder 2 at price $\frac{y_2}{x_2} \cdot p_2$. Moreover, if the bids are expressed in terms of the correlation between the quantity of items and the price (rather than the simple linear extrapolation above), there will be even more choice for the auctioneer, and, consequently, even more chance of maximising its payoff. When viewed from the bidder's perspective, the atomic nature of bids and the inability to explicitly relate price and quantity means that opportunities for trade are lost because the auctioneer may not want the entire package being offered, even though elements of it may be acceptable.

To overcome the aforementioned shortcomings associated with atomic propositions, Sandholm and Suri consider the case in which agents can submit bids that correspond to a demand or supply curve depending on whether it is an auction or a reverse auction respectively [5]. Thus, bids are expressed in terms of a curve which correlates the quantity with the price of an item. For example, an agent may express the bid as $q = 2 * p + 1$, which means that the agent is willing to trade up to $q = 2 * p + 1$ units if the unit price equals p.[2] Unfortunately, their work is limited to multi-unit single-item auctions and does not deal with the combinatorial case. This means their algorithm cannot explicitly cope with any interdependencies that may exist between the purchasing of multiple items.

In this paper, we develop novel clearing algorithms that remove the shortcomings associated with the atomic proposition nature of previous combinatorial clearing algorithms and the non-combinatorial nature of Sandholm and Suri's supply curve functions. Specifically, we consider multi-unit single-item and multi-unit combinatorial reverse auctions in which bids contain an agent's supply function. The algorithms that we develop have polynomial complexity and are shown to be within a finite bound of the optimal. For the time being, our approach is limited to reverse auctions (in which there is one buyer and multiple sellers). Nevertheless, in the future, we aim to remove this limitation and develop algorithms for forward auctions.

The remainder of the paper is organised as follows. Section 2 formalises the problem of reverse auction clearing. Section 3 developes an algorithm for the multi-unit single-item case and proves a number of properties about the algorithm. Section 4 generalises the algorithm to the multi-unit combinatorial case. Section 5 discusses related work. Section 6 concludes and presents future work.

2 REVERSE AUCTION CLEARING

This section formalises the problem of clearing in multi-unit combinatorial reverse auctions. Assume there are m items

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[2] Their price function calculates the quantity from the unit price. However, in our work, the price function will calculate the unit price from the quantity, because we find the later more natural.

(goods/services): $1, 2, ..., m$ and n bidders $a_1, a_2, ..., a_n$. The auctioneer has a demand $(q_1, q_2, ..., q_m)$, in which q_j is the quantity of item j that the auctioneer wants. Let u_i^j be the maximum quantity of item j that a_i is able or willing to provide (if a_i does not provide an item j, then $u_i^j = 0$). Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}^*$ be the set of non-negative real numbers.

The *supply function* is the price function of the items that each bidder is willing to sell. Bidder i's supply function is: $P_i : \mathbb{N}^m \to \mathbb{R}^*$, where $P_i(r_1, r_2, .., r_m)$ is the price offered by bidder i for the package of items $(r_1, r_2, ..., r_m)$ and r_j is the quantity of item j, $r_j \in \mathbb{N}$, $0 \le r_j \le u_i^j, \forall\, 1 \le j \le m$. For example, suppose that $m = 3$, then $P_1(1, 3, 2)$ will be the price agent 1 offers for a package which is composed of 1 unit of item 1, 3 units of item 2 and 2 units of item 3 altogether. We consider settings where the price function satisfies two properties:

- Discount: if $\forall j : r_j + s_j \le u_i^j$, then:
  $P_i(r_1 + s_1, r_2 + s_2, ..., r_m + s_m)$
  $$\le P_i(r_1, r_2, ..., r_m) + P_i(s_1, s_2, ..., s_m) \quad (1)$$
  That is, buying any combination of two packages altogether is cheaper than or equal to buying these two bundles separately. In game-theoretic terms, this property is also called *sub-additive* [7].
- Free Disposal: if $\forall j : 0 \le r_j \le s_j \le u_i^j$, then:
  $$P_i(r_1, r_2, ..., r_m) \le P_i(s_1, s_2, ..., s_m) \quad (2)$$
  That is, if one package has no fewer units of each item than another package, the former is not less expensive than the latter.

The above assumptions are needed for the subsequent analysis of our algorithm and we believe they are applicable to a wide range of applications (free disposal is a standard assumption adopted in most of the aforementioned work on auction clearing, sub-additivity is less frequently used but is still realistic). Thus, they do not significantly limit the scope of our results.

We now consider the *supply allocation* which is the amount the auctioneer buys from each supplier.

Definition 1 *A supply allocation is a tuple $\{r_i^j\}, 1 \le i \le n, 1 \le j \le m$ such that the auctioneer buys r_i^j units of item j from each agent a_i.*[3]

Given the definitions of the supply function and the supply allocation, the problem of reverse auction clearing is then to find a supply allocation $\{\alpha_i^j\}, 1 \le i \le n, 1 \le j \le m$ that:

- Satisfies the demand
  $$\sum_{i=1}^{n} \alpha_i^j \ge q_j, \forall 1 \le j \le m \quad (3)$$
  That is, the quantity of each item that the auctioneer buys from all bidders is not less than the auctioneer's demand for that item.
- Optimises the cost
  $$\sum_{i=1}^{n} P_i(\alpha_i^1, \alpha_i^2, .., \alpha_i^m) \text{ is minimal.} \quad (4)$$
  That is, the total price of all the units of all the items supplied by the bidders should be as small as possible.

[3] Because items are bought at the price the bidders offer, the auctioneer may buy the same package from two different bidders at different prices, i.e., the auctions have *discriminatory pricing* [5].

However, this problem has been shown to be NP-complete, even for the simplified case of single-items with piecewise linear supply curves [5].[4] Thus, it is impossible to find a polynomial algorithm, unless P = NP. Given this, a heuristic method is appropriate. To this end, the next section presents our algorithm for the single-item case (i.e. where $m = 1$), then we will deal with the combinatorial case as a generalisation in section 4.

3 MULTI-UNIT SINGLE-ITEM REVERSE AUCTIONS

Using the notation of the previous section, the multi-unit single-item case can be formulated as follows. Let q be the demand of the auctioneer and u_i be the maximum quantity of the item that a_i will provide. The supply function (in the single-item case it can be drawn as a curve, so we can call it the supply curve) is the price function of the item: $P_i : \mathbb{N} \to \mathbb{R}^*$ where $P_i(r)$ is the price for r units when they are sold altogether by bidder i.

For mathematical convenience, in this section we will use the unit price function instead of the price function. The unit price function for each bidder i is: $p_i : \mathbb{N} \to \mathbb{R}^*$ where $p_i(r)$ is the unit price for the item when r units are sold altogether by bidder i. That is, $p_i(r) = \frac{P_i(r)}{r}$.

As before, we consider settings where the supply curve satisfies the following properties:

- Discount:
  $$p_i(r) \ge p_i(s), \forall\, 0 \le r \le s \le u_i \quad (5)$$
  That is, the more units that are sold, the less the unit price is.[5]
- Free Disposal:
  $$r \cdot p_i(r) \le s \cdot p_i(s), \forall\, 0 \le r \le s \le u_i \quad (6)$$
  That is, the more units of the item that are sold, the more the total price is.

The clearing problem is then one of finding a supply allocation $\{\alpha_i\}, 1 \le i \le n$, i.e., agent a_i will provide α_i units, such that:

- The sum of supplies from bidders fulfills the auctioneer's demand:
  $$\sum_{i=1}^{n} \alpha(i) \ge q \quad (7)$$
- The total price paid by the auctioneer is minimised:
  $$\sum_{i=1}^{n} \alpha_i \cdot p_i(\alpha_i) \text{ is minimal.} \quad (8)$$

We are now in a position to express our algorithm for solving this problem. Like [6] we adopt a greedy approach for solving this problem and our algorithm is presented in Figure 1.

Theorem 1 *If there is a solution, then this algorithm will find it. That is, if the bidders can supply the units that the auctioneer demands, then this algorithm will produce an allocation. Also, the solution will supply exactly the number of units demanded by the auctioneer.*

[4] Although [5] does not explicitly consider sub-additive pricing, their proof also holds for this case.

[5] This is stronger than the Discount definition in the general case (inequation 1). We use it here because it is reasonable for single-item pricing.

Algorithm 1 *Repeat the following steps:*

- *For all i such that $u_i > q$, set $u_i = q$.*
  That is, we truncate the supply function to consider only quantities that are not bigger than the demand. This is because in order to minimise the total price, the auctioneer does not need to buy more units than its demand, since the price functions satisfy the free disposal property (inequation (6)).
- *At each step, find the bidder a_k that provides the smallest unit price, then take a_k to provide all its quantity u_k. The smallest unit price is found by determining the smallest element of the set $\{p_1(u_1), p_2(u_2), .., p_n(u_n)\}$.*
- *Repeat the steps above for the set of bidders $A \setminus a_k$ and $q_{new} = q - u_k$.*

Figure 1. The clearing algorithm for the multi-unit single-item case.

PROOF. In each step, the algorithm selects exactly one agent from the set of bidders. If its supply is less than the auctioneer's remaining demand, the algorithm takes all its supply. Otherwise it takes the quantity that is equal to the remaining demand. So, if the algorithm does not terminate beforehand, it will eventually select all bidders and take all supplies. Thus, if the bidders can supply enough units to meet the demand, the algorithm will produce an allocation. Moreover, in each step, the algorithm takes at most all the remaining demand, so the solution it produces will have the total units being equal to the auctioneer's demand. □

Theorem 2 *The complexity of algorithm 1 is $O(n^2)$.*

PROOF. At each step, it requires $O(n)$ to find the smallest element of the set $\{p_1(u_1), .., p_n(u_n)\}$. So each step has $O(n)$ complexity. As there are at most n steps, the complexity is $O(n^2)$ □

Theorem 3 *The solution generated from algorithm 1 is within a bound $b = n$ from the optimal. That is, let $P_n(O)$ be the optimal total price and $P_n(S)$ be the total price of the solution of the algorithm. Then:*

$$\frac{P_n(S)}{P_n(O)} \leq n \tag{9}$$

PROOF. We prove by induction of the number of bidders n.

Base case ($n = 1$): In the case where $n = 1$ the solution is optimal (because we have only one bid) so (9) is true with $n = 1$.

Inductive step: Suppose that (9) is true for n, we will prove that (9) is also true for $n+1$. That is, let $(r_1, r_2, ..., r_{n+1})$ be the supply allocation that the algorithm generates. Then we have to prove that: $\sum_{i=1}^{n+1} r_i \cdot p_i(r_i) \leq (n+1) \cdot P_{n+1}(O)$. Or equivalently, for all other supply allocations $(t_1, t_2, .., t_{n+1})$ that satisfy the demand, their total price is greater than $\frac{1}{n+1}$ times the total price of the supply allocation produced by algorithm 1. That is, $\forall\, t_1, t_2, .., t_{n+1}$ such that: $0 \leq t_i \leq u_i, \forall 1 \leq i \leq n+1$ and $\sum_{i=1}^{n+1} t_i \geq q$, then: $\sum_{i=1}^{n+1} r_i \cdot p_i(r_i) \leq (n+1) \cdot (\sum_{i=1}^{n+1} t_i \cdot p_i(t_i))$.

Proof of inductive step

Without loss of generality, assume that agent a_{n+1} provides the smallest unit price, that is, $p(u_{n+1}) = min_{i=1}^{n+1} p(u_i)$. This means that agent a_{n+1} is selected in the first step of algorithm 1 and:

$$r_{n+1} = u_{n+1} \tag{10}$$

Because supply allocation $\{t_i\}$ satisfies the demand (as in (7)):

$$\sum_{i=1}^{n+1} t_i \geq q \tag{11}$$

But supply allocation $\{r_i\}$ supplies exactly the demand quantity (by Theorem 1), thus: $\sum_{i=1}^{n+1} r_i = q \Rightarrow \sum_{i=1}^{n+1} t_i \geq \sum_{i=1}^{n+1} r_i$ $\Rightarrow \sum_{i=1}^{n} t_i \geq \sum_{i=1}^{n} r_i$ (as $t_{n+1} \leq u_{n+1} = r_{n+1}$, from (10))

Moreover, by inductive hypothesis, (9) is true for n agents.

$$\Rightarrow \sum_{i=1}^{n} r_i \cdot p_i(r_i) \leq n \cdot \sum_{i=1}^{n} t_i \cdot p_i(t_i) \leq n \cdot \sum_{i=1}^{n+1} t_i \cdot p_i(t_i) \tag{12}$$

(as $t_{n+1} \cdot p_{n+1}(t_{n+1}) \geq 0$)

Also: $r_{n+1} \leq q \leq \sum_{i=1}^{n+1} t_i$ (from (11))

$$\Rightarrow r_{n+1} \cdot p_{n+1}(r_{n+1}) \leq \sum_{i=1}^{n+1} t_i \cdot p_{n+1}(r_{n+1})$$

$$\Rightarrow r_{n+1} \cdot p_{n+1}(r_{n+1}) \leq \sum_{i=1}^{n+1} t_i \cdot p_i(t_i) \tag{13}$$

(as $p_{n+1}(r_{n+1})$ is the smallest unit price)

From (12) and (13), we have: $\sum_{i=1}^{n+1} r_i \cdot p_i(r_i) \leq (n+1) \cdot (\sum_{i=1}^{n+1} t_i \cdot p_i(t_i))$

The completion of the inductive step completes our proof. □

Although multi-unit single-item auctions are not our main target case, this algorithm still represents a novel contribution in its own right. While [5] targets the same environment as this, the algorithms are only applicable in the specific case where the supply curves are linear. In contrast, our result is applicable to the more general case; that is, sub-additive, free disposal supply curves.

4 MULTI-UNIT COMBINATORIAL REVERSE AUCTIONS

To deal with the multi-unit combinatorial case, we need to add one more assumption about the price functions of the items, namely there exists a number $K > 1$ such that for any price function from any bidder, K units of any item will be more expensive than 1 unit of any other item: $\forall\ 1 \leq j, k \leq m, j \neq k, d \in \mathbb{N}$:

$$P_i(r_1, .., r_j + d, .., r_k, .., r_m) \leq P_i(r_1, .., r_j, .., r_k + Kd, .., r_m) \tag{14}$$

That is, for any package, if we substitute d unit of any item in this package by $K \cdot d$ units of any other item, then the price of the new package will be more expensive or equal to the price of the old package. We believe this is a realistic assumption because in a competitive market the unit price of any item is always likely to be within a finite range; that is, it cannot be arbitrarily high or low.

From this, a number of lemmas follow:

Lemma 1 *For any package of items, if we replace d units of any item with d units of any other item, then the total price of the new package of items is not bigger than K times the total price of the old package: $\forall\ 1 \leq j, k \leq m, j \neq k, d \in \mathbb{N}$:*

$$P_i(r_1, .., r_j + d, .., r_k, .., r_m) \leq K \cdot P_i(r_1, .., r_j, .., r_k + d, .., r_m) \tag{15}$$

PROOF. We have: $P_i(r_1, .., r_j + d, .., r_k, .., r_m) \leq P_i(r_1, .., r_j, .., r_k + Kd, .., r_m)$ (by (14))

Also P_i satisfies the free disposal property (in (2)), thus:
$P_i(r_1, .., r_j + d, .., r_k, .., r_m)$
$\leq P_i(Kr_1, .., Kr_j, .., Kr_k + Kd, .., Kr_m)$
$\leq K \cdot P_i(r_1, .., r_j, .., r_k + d, .., r_m)$ (by (1)) □

Lemma 2 *For any two packages, if the total number of units of the first package is not bigger than the total number of units of the second package, then the total price of the first package is not bigger than K^{m-1} times the total price of the second package: $\forall 1 \leq i \leq n$, $\forall r_1, r_2, ..., r_m, s_1, s_2, ..., s_m$ such that $\sum_{j=1}^{m} r_j \leq \sum_{j=1}^{m} s_j$, then:*

$$P_i(r_1, r_2, ..., r_m) \leq K^{m-1} P_i(s_1, s_2, ..., s_m) \quad (16)$$

PROOF. Let $d_j = (s_j - r_j)$

$$\Rightarrow \sum_{j=1}^{m} d_j = \sum_{j=1}^{m} s_j - \sum_{j=1}^{m} r_j \geq 0 \quad (17)$$

Now there are 2 cases:

- *Case 1*: $d_i \geq 0, \forall 1 \leq i \leq m$. Then we have: $P_i(r_1, r_2, ..., r_m) \leq P_i(s_1, s_2, ..., s_m)$ (because P_i satisfies the free disposal property in (2)). Thus $P_i(r_1, r_2, ..., r_m) \leq K^{m-1} P_i(s_1, s_2, ..., s_m)$.
- *Case 2*: There exists a $d_k < 0$. Then without loss of generality, suppose that $d_m < 0$. We have: $P_i(r_1, r_2, ..., r_{m-1}, r_m) \leq K \cdot P_i(r_1 - d_m, r_2, ..., r_{m-1}, s_m)$ (by lemma 1).
  Let $r_1^{(2)} = r_1 - d_m$, $r_i^{(2)} = r_i, \forall\, 2 \leq i \leq m-1$.
  $\Rightarrow P_i(r_1, r_2, ..., r_{m-1}, r_m) \leq K \cdot P_i(r_1^{(2)}, r_2^{(2)}, ..., r_{m-1}^{(2)}, s_m)$
  Also: $\sum_{j=1}^{m-1} r_j^{(2)} = \sum_{j=1}^{m-1} r_j - d_m \Rightarrow \sum_{j=1}^{m-1} r_j^{(2)} \leq \sum_{j=1}^{m-1} s_j$ (by (17)).

Repeat the whole step above, it will take at most $m-1$ steps to terminate. Thus, after at most $m-1$ steps, we will have: $P_i(r_1, r_2, ..., r_m) \leq K^{m-1} P_i(s_1, s_2, ..., s_m)$ □

Lemma 3 *For any two packages, if the total number of units of the first package is not bigger than the total number of units of the second package, then the average unit price of the first package is not smaller than $\frac{1}{2K^{m-1}}$ times the average unit price of the second package: $\forall 1 \leq i \leq n$, $r_1, r_2, ..., r_m, s_1, s_2, ..., s_m$ such that $\sum_{j=1}^{m} r_j \leq \sum_{j=1}^{m} s_j$, then:*

$$2K^{m-1} \cdot \frac{P_i(r_1, r_2, ..., r_m)}{r_1 + r_2 + ... + r_m} \geq \frac{P_i(s_1, s_2, ..., s_m)}{s_1 + s_2 + ... + s_m} \quad (18)$$

PROOF. Let $k = [\frac{\sum_{j=1}^{m} s_j}{\sum_{j=1}^{m} r_j}]$, that is, k is the integral part of $\frac{\sum_{j=1}^{m} s_j}{\sum_{j=1}^{m} r_j}$.

$$\Rightarrow k \leq \frac{\sum_{j=1}^{m} s_j}{\sum_{j=1}^{m} r_j} < k+1 \quad (19)$$

$$\Rightarrow (k+1) \sum_{j=1}^{m} r_j > \sum_{j=1}^{m} s_j$$

$$\Rightarrow K^{m-1} P_i((k+1)r_1, .., (k+1)r_m) \geq P_i(s_1, .., s_m)$$

(by lemma 2)

$$\Rightarrow K^{m-1}(k+1) P_i(r_1, .., r_m) \geq P_i(s_1, .., s_m) \text{ (by (1))} \quad (20)$$

Also: $\sum_{j=1}^{m} r_j \leq \sum_{j=1}^{m} s_j \Rightarrow \frac{\sum_{j=1}^{m} s_j}{\sum_{j=1}^{m} r_j} \geq 1$
$\Rightarrow k \geq 1 \Rightarrow k+1 \leq 2k \leq 2 \cdot \frac{\sum_{j=1}^{m} s_j}{\sum_{j=1}^{m} r_j}$ (from (19))
$\Rightarrow 2K^{m-1} \cdot \frac{\sum_{j=1}^{m} s_j}{\sum_{j=1}^{m} r_j} P_i(r_1, .., r_m) \geq P_i(s_1, .., s_m)$ (by (20))
$\Rightarrow 2K^{m-1} \cdot \frac{P_i(r_1, r_2, ..., r_m)}{r_1 + r_2 + ... + r_m} \geq \frac{P_i(s_1, s_2, ..., s_m)}{s_1 + s_2 + ... + s_m}$ □

With these lemmas in place, we can now present the generalisation of the single-item algorithm to the combinatorial case (Figure 2).

Algorithm 2 *Repeat the following steps:*

- *For all i, j such that $u_i^j > q_j$, set $u_i^j = q_j$.*
  That is, we truncate the supply function to consider only quantities that are not bigger than the demand. This is because to minimise the total price, the auctioneer does not need to buy more units than its demand, as the price functions satisfy the free disposal property (in (2)).
- *Find the bidder a_k such that:*

$$\frac{P_k(u_k^1, u_k^2, ..., u_k^m)}{u_k^1 + u_k^2 + ... + u_k^m} \text{ is minimal,}$$

  then select a_k to provide all its units $(u_k^1, u_k^2, ..., u_k^m)$.
  That is, we consider all the biggest packages offered by the bidders, then choose the package that offers the lowest average unit price.
  Note that this is not necessarily the package that offers the lowest average in all packages, because a smaller package may have a smaller average unit price.
- *Repeat the steps with the new set of bidders $A \setminus a_k$ and demand $q_j^{new} = q_j - u_k^j$.*

Figure 2. The clearing algorithm for the multi-unit combinatorial case.

We can now analyse this algorithm to assess its properties.

Theorem 4 *If there is a solution, then this algorithm will find it. That is, if the bidders can supply the units that the auctioneer demands, then this algorithm will produce an allocation. Also, the solution supplies exactly the number of units demanded by the auctioneer.*

PROOF. The proof is the same as that of theorem 1. □

Theorem 5 *The complexity of algorithm 2 is $O(n^2)$*

PROOF. At each step, it requires $O(n)$ to find the smallest element of the set $\{\frac{P_k(u_k^1, u_k^2, ..., u_k^m)}{u_k^1 + u_k^2 + ... + u_k^m}\}_{i=1}^{n}$. So each step has $O(n)$ complexity. As there are at most n steps, the complexity is $O(n^2)$. □

Theorem 6 *The solution generated from algorithm 2 is within a bound $b = 2n \cdot K^{m-1}$ from the optimal. That is, let $P_n(O)$ be the optimal total price and $P_n(S)$ be the total price of the solution of the algorithm. Then:*

$$\frac{P_n(S)}{P_n(O)} \leq 2n \cdot K^{m-1} \quad (21)$$

PROOF. We prove by induction of the number of bidders n.

Base case ($n = 1$): In this case, the solution is optimal (because there is only one bid to choose from), so (21) is true with $n = 1$.

Inductive step: Suppose that (21) is true for n, we will prove that (21) is also true for $n+1$. That is, let $\{r_i^j\}, 1 \leq i \leq n+1, 1 \leq j \leq m$ be the supply allocation that algorithm 2 generates. Then we have to prove that: $\sum_{i=1}^{n+1} P_i(r_i^1, r_i^2, ..., r_i^m) \leq 2n \cdot K^{m-1} P_{n+1}(O)$.

Or equivalently, for every other supply allocation $\{t_i^j\}$ that satisfies the auctioneer's demand, the total price of $\{r_i^j\}$ is not bigger than $2n \cdot K^{m-1}$ times the total price of $\{t_i^j\}$: $\sum_{i=1}^{n+1} P_i(r_i^1, r_i^2, ..., r_i^m) \leq 2(n+1) \cdot K^{m-1} \sum_{i=1}^{n+1} P_i(t_i^1, t_i^2, ..., t_i^m)$.

Proof of inductive step

Without loss of generality, assume that agent a_{n+1} provides the lowest average price in all the biggest packages:

$$\frac{P_{n+1}(u_{n+1}^1, u_{n+1}^2, ..., u_{n+1}^m)}{u_{n+1}^1 + u_{n+1}^2 + ... + u_{n+1}^m} = \min_{i=1}^{n+1} \frac{P_i(u_i^1, u_i^2, ..., u_i^m)}{u_i^1 + u_i^2 + ... + u_i^m} \quad (22)$$

That means a_{n+1} is selected in the first step of the algorithm and:

$$r_{n+1}^j = u_{n+1}^j, \text{ for all } 1 \leq j \leq m \quad (23)$$

For all $1 \leq j \leq m$, because supply allocation $\{t_i^j\}$ satisfies the auctioneer's demand (inequation (3)):

$$\Rightarrow \sum_{i=1}^{n+1} t_i^j \geq q_j \quad (24)$$

But supply allocation $\{r_i^j\}$ supplies exactly the demand quantity (by Theorem 4) $\Rightarrow \sum_{i=1}^{n+1} r_i^j = q_j \Rightarrow \sum_{i=1}^{n+1} t_i^j \geq \sum_{i=1}^{n+1} r_i^j$
$\Rightarrow \sum_{i=1}^{n} t_i^j \geq \sum_{i=1}^{n} r_i^j$ (as $t_{n+1}^j \leq u_{n+1}^j = r_{n+1}^j$ by (23))

Moreover, by inductive hypothesis, (21) is true for n agents
$\Rightarrow \sum_{i=1}^{n} P_i(r_i^1, r_i^2, ..., r_i^m) \leq 2nK^{m-1} \cdot \sum_{i=1}^{n} P_i(t_i^1, t_i^2, ..., t_i^m)$

$$\Rightarrow \sum_{i=1}^{n} P_i(r_i^1, r_i^2, ..., r_i^m) \leq 2nK^{m-1} \cdot \sum_{i=1}^{n+1} P_i(t_i^1, t_i^2, ..., t_i^m) \quad (25)$$

(because $P_{n+1}(t_{n+1}^1, t_{n+1}^2, ..., t_{n+1}^m) \geq 0$)

Also: $P_{n+1}(r_{n+1}^1, r_{n+1}^2, ..., r_{n+1}^m)$
$= P_{n+1}(u_{n+1}^1, u_{n+1}^2, ..., u_{n+1}^m)$ (by (23))
$= (\sum_{j=1}^{m} u_{n+1}^j) \cdot \frac{P_{n+1}(u_{n+1}^1, u_{n+1}^2, ..., u_{n+1}^m)}{\sum_{j=1}^{m} u_{n+1}^j}$

But $u_{n+1}^j \leq q_j, \forall 1 \leq j \leq m. \Rightarrow P_{n+1}(r_{n+1}^1, r_{n+1}^2, ..., r_{n+1}^m)$
$\leq (\sum_{j=1}^{m} q_j) \frac{P_{n+1}(u_{n+1}^1, u_{n+1}^2, ..., u_{n+1}^m)}{\sum_{j=1}^{m} u_{n+1}^j}$
$\leq (\sum_{j=1}^{m} \sum_{i=1}^{n+1} t_i^j) \frac{P_{n+1}(u_{n+1}^1, u_{n+1}^2, ..., u_{n+1}^m)}{\sum_{j=1}^{m} u_{n+1}^j}$ (by (24))
$\leq \sum_{i=1}^{n+1} (\sum_{j=1}^{m} t_i^j \frac{P_i(u_i^1, u_i^2, ..., u_i^m)}{\sum_{j=1}^{m} u_i^j})$ (because of (22))
$\leq \sum_{i=1}^{n+1} (\sum_{j=1}^{m} t_i^j 2K^{m-1} \frac{P_i(t_i^1, t_i^2, ..., t_i^m)}{\sum_{j=1}^{m} t_i^j})$ (by lemma 3)

$$\Rightarrow P_{n+1}(r_{n+1}^1, r_{n+1}^2, ..., r_{n+1}^m) \leq 2K^{m-1} \cdot (\sum_{i=1}^{n+1} P_i(t_i^1, t_i^2, ..., t_i^m)) \quad (26)$$

From (25) and (26) we have: $\sum_{i=1}^{n+1} P_i(r_i^1, r_i^2, ..., r_i^m) \leq 2(n+1) \cdot K^{m-1} \sum_{i=1}^{n+1} P_i(t_i^1, t_i^2, ..., t_i^m)$

The completion of the inductive step completes our proof. □

5 RELATED WORK

As already discussed, most of the previous work on clearing algorithms for combinatorial auctions has been based on atomic proposition auctions. So by removing this restriction, our algorithm produces more efficient allocations. In particular, Sandholm et. al. [6] have categorised and analysed the complexity of various kinds of atomic proposition types (e.g. auctions, reverse auctions, exchanges). They showed that clearing combinatorial atomic proposition auctions is NP-complete, even for the simple case of single-units (i.e. each item has only one unit). Thus, heuristic methods are typically used to tackle this problem.

In more detail, for single-unit combinatorial auctions, Nisan [4] showed that Linear Programming can produce the optimal solution in a reasonable time in some specific cases (e.g. linear order bids, mutual exclusion bids and substructure bids), and suggested using greedy and Branch-and-Bound algorithms based on Linear Programming for the other cases. However, in our view, this Linear Programming-based approach cannot easily be applied to our situation because the problem of clearing auctions with supply/demand functions cannot easily be modeled. Other researchers such as Gonen and Lehmann [2] and Leyton Brown et. al. [3] have further investigated the use of Branch-and-Bound techniques to solve the clearing problem. Unfortunately, however, these Branch-and-Bound algorithms cannot guarantee to produce the optimal solution in polynomial time.

In contrast to the above, Sandholm and Suri [5] considered multi-unit single-item auctions with bids in the form of supply/demand curves of some specific types (linear and piecewise linear curves). However, as discussed in section 1, this work does not deal with multi-unit combinatorial auctions.

6 CONCLUSIONS AND FUTURE WORK

In this paper we provided, for the first time, polynomial algorithms for clearing multi-unit combinatorial reverse auctions with supply functions. While previous work has concentrated on single-item auctions with supply/demand curves or combinatorial auctions with atomic propositions, we generalised the problem to multi-unit single-item and multi-unit combinatorial auctions with supply functions. For this very general case, we showed that our algorithms are of polynomial complexity and can generate solutions that are within a bound of the optimal. We believe this generalisation is an important step towards realising the full application potential of combinatorial auctions since it enables us to deal with a maximally flexible and efficient scheme in a computationally tractable manner.

For the future, we aim to reduce the bound from the optimal within this framework or to prove the optimality of the error bound. Moreover, we aim to extend our algorithms so that they are also applicable to the forward case. We also aim to develop the algorithms for particular classes of domain in which stronger assumptions can be made about the supply and supply allocation functions in order to find better approximations for these more specific cases.

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