The strength of Ramsey Theorem for coloring relatively large sets

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Outline

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1. Motivation and Background
   - Ramsey Theorem for large sets
   - **RT**$(!\omega)$ and Ramsey's Theorem
   - Known facts about Ramsey Theorems
2. Lower Bounds on **RT**$(!\omega)$
   - Weak Lower Bound on **RT**$(!\omega)$
   - Strong Lower Bounds on **RT**$(!\omega)$
3. Upper Bounds on **RT**$(!\omega)$
4. Conclusion and prospects

Overview

We analyze the effective and proof-theoretic strength of an infinitary Ramsey-type theorem due to Pudlák and Rödl and independently to Farmaki.

The main result is that the theorem is equivalent over Computable Mathematics ($\mathbf{RCA}_0$) to closure under the ω-th Turing jump.

Large Sets

- A set $S = \{s_0, s_1, \ldots, s_n\}$ is
  - *large* if $n \geq s_0$, and is
  - *exactly large* if $n = s_0$.
- The notion of *large set* was introduced by Paris and Harrington and is the key notion in the Paris-Harrington Large Ramsey Theorem.
- We denote by $[X]^{!\omega}$ the set of exactly large subsets of X.

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Coloring Large Sets

The theorem we study is the following.

Theorem

For every infinite $M \subseteq \mathbf{N}$, for every coloring C of the exactly large subsets of $\mathbf{N}$ in two colors there exists an infinite homogeneous set $H \subseteq M$.

Alternative formulations:

1. Pudlák-Rödl: uniform families.
2. Farmaki: (thin) Schreier families (Banach Space Theory).
   Schreier family:

$$\{s = \{n_1, \ldots, n_k\} : n_1 \geq k\}$$

   Thin Schreier family: $n_1 = k$.

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A Combinatorial Proof of $\mathbf{RT}(!\omega)$

Let M be an infinite subset of $\mathbf{N}$, let $C : [\mathbf{N}]^{!\omega} \to 2$. We build an infinite homogeneous subset $L \subseteq M$ for C in stages. Let $C_a \colon [\mathbf{N} \setminus \{1, \ldots, a\}]^a \to 2$ be defined by

$$C_a(x_1, \ldots, x_a) = C(a, x_1, \ldots, x_a).$$

We define a sequence $\{(a_i, X_i)\}_{i \in \mathbf{N}}$ such that

- $a_0 = \min(M)$,
- $X_{i+1} \subseteq X_i \subseteq M$,
- X_i is an infinite and C_{a_i}–homogeneous and $a_i < \min(X_i)$,
- $a_{i+1} = \min X_i$.

At the i-th step of the construction use Ramsey's Theorem for coloring a_i–tuples from the infinite set X_{i-1} (where $X_{-1} = M$).
Finally apply the Infinite Pigeonhole Principle to the sequence $\{a_i\}_{i \in \mathbf{N}}$ to get an infinite C–homogeneous set.

Outline

$\mathbf{RT}(!\omega)$ and $\forall n \mathbf{RT}^n_2$

Proposition

Over $\mathbf{RCA}_0$, $\mathbf{RT}(!\omega)$ *implies* $\forall n \mathbf{RT}^n_2$.

Let $n \geq 1$ and $C : [\mathbf{N}]^n \to 2$ be given. Define $C' : [\mathbf{N}]^{!\omega} \to 2$ as follows. Let $S = \{s_0, \ldots, s_m\}$ be an exactly large set (then $m = s_0$).

$$C'(s) := \begin{cases} C(s_0, \ldots, s_{n-1}) & \text{if } s_0 \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

1. Let H be an infinite C'-homogeneous set of colour $i \in \{0, 1\}$.
2. Let $H' = H \cap [n, \infty)$, and let $S \in [H']^n$. Then $\min(S) \geq n$.
3. Let S' be any exactly large set extending S in H'.
4. Then $C(S) = C'(S') = i$. Thus H' is C-homogeneous of color i.

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Computability results

Theorem (Jockusch 1972)

1. *For each* $n \geq 2$ *there exists a computable coloring* $C : [\mathbf{N}]^n \to 2$ *with no infinite homogeneous set in* Σ^0_n.
2. *For each* n*, for each computable coloring* $C : [\mathbf{N}]^n \to 2$*, there exists an infinite* C*-homogeneous set in* Π^0_n.
3. *For each* $n \geq 2$ *there exists a computable coloring* $C : [\mathbf{N}]^n \to 2$ *all of whose homogeneous sets compute* $0^{(n-2)}$.

Reverse Mathematics results

Theorem (Simpson)

The following are equivalent over $\mathbf{RCA}_0$*.*

1. $\mathbf{RT}^3$,
2. $\mathbf{RT}^n$ *for any* $n \in \mathbf{N}$, $n \geq 3$,
3. $\forall X \exists Y (Y = X')$.

Thus $\mathbf{RCA}_0 + \mathbf{RT}^3 \equiv \mathbf{ACA}_0$.

Theorem (McAloon 1985)

The following are equivalent over $\mathbf{RCA}_0$*.*

1. $\forall n \mathbf{RT}^n$,
2. $\forall n \forall X \exists Y (Y = X^{(n)})$.

Thus $\mathbf{RCA}_0 + \forall n \mathbf{RT}^n \equiv \mathbf{ACA}_0'$.

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A weak Lower Bound

Theorem

There exists a computable coloring $C\colon [\mathbf{N}]^{!\omega} \to 2$ such that any infinite homogeneous set for C is not Σ^0_i, for any $i \in \mathbf{N}$.

Recall that there exists sets that are incomparable with all $0^{(i)}$, $i \geq 1$.

Proposition

There exists a computable sequence of X-computable functions $e^X_n\colon [\mathbf{N}]^{n+2} \to \{0,1\}$ such that for any $n \geq 0$, for every $i \in \mathbf{N}$,

1. *$e^{K_i}_n$ is K_i-computable, and*
2. *$e^{K_i}_n$ computes a coloring with no homogeneous set in Σ^0_{i+n+2}.*

Proof of Theorem from Proposition

Define $C : [\mathbf{N}]^{!\omega} \to 2$ as follows.

$$C(s_0, s_1, \ldots, s_{s_0}) = e^{K_0}_{s_0-2}(s_1, \ldots, s_{s_0}).$$

If Y is ihs for C then for every $a \in Y$, $Y \cap \{x \in \mathbf{N} \,:\, x \geq a\}$ is ihs for $e^{K_0}_{a-2}$. Then it is not Σ^0_a by Proposition.

Proof of Proposition

Ingredients:

1. Jockusch's result: There exists a computable $e : [\mathbf{N}]^2 \to 2$ without ihs in Σ^0_2.
2. Relativizes to: There exists an X-computable $e^X : [\mathbf{N}]^2 \to 2$ without ihs in Σ^0_{i+2} if X is Σ^0_i-complete.
3. Schoenfield's Limit Lemma: functions computable in K are lim-computable.
4. Generalizes to: If B is c.e. in A and f is computable in B then f is A-lim-computable.
5. Uniform formulation: there exists a function $g^X(i, e, x, s)$ such that, if $B = W_i^A$ and $f = \{e\}^B$ then

$$f(x) = \lim_{s \to \infty} g^A(i, e, x, s)$$

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Proof of Proposition (continued)

1. Fix $\{K_i\}_{i \in \mathbf{N}}$ s.t. $K_0 = \varnothing$, K_{i+1} is Σ^0_{i+1}-complete.
2. Base: result of Jockusch's, relativized.
3. Step: define $e^X_{n+1} : [N]^{n+3} \to \{0, 1\}$. Ensure
   1. If $X = K_i$ then: homset for $e^{K_i}_{n+1} \Rightarrow$ homset for $e^{K_{i+1}}_n$.
   2. Index for e^X_{n+1} obtained effectively from index for e^X_n.
4. By Shoenfield's Lemma we have that

$$\lim_{s \to \infty} g^{K_i}(\overline{e_n}, x_1, \ldots, x_{n+2}, s) = e^{K_{i+1}}_n(x_1, \ldots, x_{n+2}).$$

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Define e^X_{n+1} as follows.

$$e^X_{n+1}(x_1, \dots, x_{n+2}, s) := g^X(\overline{e_n}, x_1, \dots, x_{n+2}, s).$$

If Y is an infinite homset for $e^{K_i}_{n+1}$ colored $b \in \{0, 1\}$, then any tuple $(x_1, \dots, x_{n+2}) \in [Y]^{n+2}$ has to be colored b by $e^{K_{i+1}}_n$

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Strong Lower Bounds

Theorem

For each set A there exists a computable in A coloring $C_\omega : [\mathbf{N}]^{!\omega} \to 2$ *such that all infinite homogeneous sets for* C_ω *compute* $A^{(\omega)}$.

Proposition

There exists computable in A colorings $C_n : [\mathbf{N}]^{n+1} \to \{0, 1\}$, *for* $n \in \mathbf{N}$ *and* $n \geq 2$, *and TMs* $M_n(x, y)$ *s.t., for any* $n \geq 2$, *the following three points hold.*

1. *All infinite homogeneous sets for* C_n *have color* 1.
2. *If X is an ihs for* C_n *then for any* $a_1 < \cdots < a_{n+1} \in X$ *then* $M_n(x, (a_1, \ldots, a_{n+1}))$ *decides* $A^{(n-1)}$ *for machines with indices less than or equal to* a_1.
3. *Machines* M_n *are total. (If their inputs are not from an ihs for* C_n *then we have no guarantee on the correctness of their output).*

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Proof of Theorem from Proposition

Define

$$C_\omega(a_1, \dots, a_{a_1+1}) = C_{a_1-1}(a_2, \dots, a_{a_1+1}).$$

Claim: Any ihs H for C_ω computes $A^\omega = \{(i,j) : j \in A^i\}$.

- For any $h \in H$, C_ω coincides with C_{h-1} on large sets with minimum h and rest in $H^- = H - \{1, \dots, h\}$.
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Proof of Proposition

For this proof let

$$X' = \{e : \{e\}^X(0)\downarrow\}.$$

We define C_2 as

$$C_2(k, y, z) = \begin{cases} 1 & \text{if } \forall e \leq k(\{e\}_y^A(0)\downarrow \Leftrightarrow \{e\}_z^A(0)\downarrow) \\ 0 & \text{otherwise.} \end{cases}$$

1. Any infinite homset has color 1.
2. We can build machine $M_2(e, (k, b, b'))$ deciding A' up to k if $b' > b > k$ are from an infinite homset for C_2.

Proof of Proposition, continued

We set $C_{n+1}(a_1, \ldots, a_{n+2}) =$

$$\begin{cases} 1 & \text{if } \{a_1, \ldots, a_{n+2}\} \text{ is } C_n\text{–homogeneous and} \\ & \forall e \leq a_1(\{e\}^Y_{a_2}(0){\downarrow} \Leftrightarrow \{e\}^Y_{a_3}(0){\downarrow}), \text{ where} \\ & Y = \{i \leq a_2 : M_n(i, (a_2, \ldots, a_{n+2})) \text{ accepts,}\} \\ 0 & \text{otherwise.} \end{cases}$$

We are approximating the condition

$$\forall e \leq a_1(\{e\}^{A^{(n-1)}}_{a_2}(0){\downarrow} \Leftrightarrow \{e\}^{A^{(n-1)}}_{a_3}(0){\downarrow})$$

Using homegeneity and infinity we can get

$$\forall e \leq a_1(\{e\}^Y_{a_2}(0){\downarrow} \Leftrightarrow \{e\}^{A^{(n-1)}}(0){\downarrow}),$$

for suitable a_1, a_2.

Reverse Mathematics Corollary

Theorem

$\mathbf{RT}(!\omega)$ *implies* $\forall X \exists Y (Y = X^{(\omega)})$ *over* $\mathbf{RCA}_0$.

Alternative Proof of Theorem

We use the following Proposition, which can be proved by adapting an argument from a recent paper by Dzhafarov and Hirst (2011).

Proposition

For every set X there exists a computable in X coloring $C^X : [\mathbf{N}]^{!\omega} \to 2$ such that if $H \subseteq 2\mathbf{N}$ is an infinite homogeneous set for C then H computes $X^{(n-1)}$ for every $2n \in H$.

Proof of Theorem from Proposition

- Recall $\{0^i : i \in \mathbf{N}\}$ has no lub.
- Enderton-Putnam result: If I is infinite such that $(\forall i \in I)(X^i \leq_T Y)$, then $X^\omega \leq_T Y^2$.
- Prove Theorem as follows.
  1. X computable, C^X as in Proposition, H ihs for C^X.
  2. Then for infinitely many $i \in \mathbf{N}$, $X^i \leq_T H$.
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Proof of Proposition

$$Y = X' \Leftrightarrow \forall x \forall e(\langle x, e\rangle \in Y \leftrightarrow \exists s(\{e\}^X_s(x)\downarrow)).$$

For any set X and integer s define

$$X'_s = \{\langle m, e\rangle \ : \ (\exists t < s) m \in W^X_{e,t}\}.$$

For integers $u_1, \dots, u_n$ and s define $X^{(0)} = X$, and

$$X^{(n+1)}_{u_n,\dots,u_1,s} = (X^{(n)}_{u_n,\dots,u_1})'_s.$$

Let $A = \{a_0, \dots, a_{2n+1}\}$ be exactly large. Let $C^X(A) = 1$ if there exist $1 \leq i \leq n$ and $\exists (e, m) < a_{n-i}$ such that

$$\neg((m, e) \in X^{(i)}_{a_n,\dots,a_{n-i+1}} \leftrightarrow (m, e) \in X^{(i)}_{a_{2n},\dots,a_{2n-i+1}})$$

and $C^X(A) = 0$ otherwise.

Let $H \subseteq [2, \infty) \cap 2\mathbf{N}$ be an ihs for C^X. We claim that

1. The color of C^X on $[H]^{!\omega}$ is 0.
2. For every $h \in H$, $X^{(n-1)}$ is definable by recursive comprehension from H, where $h = 2n$.

A proof in $\mathbf{ACA}_0^+$

Theorem

$\forall X \exists Y (Y = X^{(\omega)})$ implies $\mathbf{RT}(!\omega)$ *over* $\mathbf{RCA}_0$.

Idea of proof:

- Turn the induction in the combinatorial proof of $\mathbf{RT}(!\omega)$ into a first-order induction.
- Replace the sets X_i by Turing machines with oracles from $C^{(a)}$, (for a an element of a model of $\mathbf{RCA}_0$) constructed in a uniform way and computing the sets X_i.

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We want the following first-order induction: for each n there exists a sequence $\{(a_i, f_{a_i}) : i \leq n\}$ such that:

- $a_0 = 2$,
- $rg(f_{a_{i+1}}) \subseteq rg(f_{a_i}) \subseteq M$,
- $rg(f_{a_i})$ is infinite and C_{a_i}-homogeneous,
- $a_{i+1} = min(rg(f_{a_i}) \cap \{x \in \mathbf{N} : x > a_i\}$.

Lemma

Let $a \geq 1$. Let $C\colon [U]^a \to 2$. One can find effectively a machine f_a with oracle $(C \oplus U)^{(2a)}$ such that f_a computes a C–homogeneous set.

1. Base: f_1. Ask the $\Pi^0_2(C \oplus U)$ oracle whether $\forall n \exists k \geq n(C(k) = 0 \wedge U(k))$. If yes, then compute the set $C(x) = 0 \wedge U(x)$, else compute the set $C(x) = 1 \wedge U(x)$.
2. Step: f_{a+1}.
    1. Build Erdős–Rado tree T_a for $C\colon [U]^{a+1} \to 2$ (computably in $C \oplus U$).
    2. Obtain index for a machine computing the leftmost infinite path P of T_a using a $\Pi^0_2(C \oplus U)$–complete oracle.
    3. The color of any $(a+1)$–tuple from P does not depend on the last element of the tuple. Induces C' coloring of a-tuples.
    4. Use f_a (inductive hypothesis) with oracle $(C' \oplus P)^{(2a)}$. Any infinite C'–homogeneous subset of P computed by f_a is also C–homogeneous.
    5. Since P is recursive in $\Pi^0_2(C \oplus U)$, the complexity of the oracle is $(C \oplus U)^{(2(a+1))}$.

Conclusion and prospects

1. Other results:
   1. Analogous results for Regressive Ramsey Theorem on large sets.
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