On the Worm Principle

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Outline

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- I present some elementary observations relating the Beklemishev's Worm Principle to Kirby-Paris' Hydra Game and to Friedman's gap-embedding.
- Prompted by these observations Beklemishev proved much more interesting characterization results!

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The Worm Principle

- Beklemishev extracted a finite **PA**-independence result from his provability-logic ordinal analysis, the Worm Principle.
- A worm is a function $f : [0, n] \to \mathbf{N}$.
- $next(w, m)$: If $f(n) = 0$ then $next(w, m) = (f(0), \ldots, f(n-1))$. Else, let $k = \max(i < n : f(i) < f(n))$. Let $r = (f(0), \ldots, f(k))$ and $s = (f(k+1), \ldots, f(n-1), f(n) - 1)$. Then $next(w, m) = rs \ldots s$ with $m + 1$ copies of s.
- Given w, let $w_0 = w$ and $w_{n+1} = next(w_n, n+1)$.
- EWD is the following Π^0_2 sentence in the language of **PA**: $\forall w \exists n (w_n = \varnothing)$.
- Beklemishev: EWD is equivalent (over **PRA**) to the 1-consistency of **PA**, and therefore unprovable in **PA**.

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Hydra Game

- The Hydra Game is defined on finite rooted trees.
- Rule: See picture!
- Kirby and Paris: Termination of the Hydra Game is unprovable in Peano Arithmetic.
- The Hydra Game can be also presented as the iteration of fundamental sequences on ordinal notations:

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WP reduces to HG

- The Worm Principle can be reduced to Kirby-Paris' Hydra Game.
- The reduction can be done using an explicit combinatorial map $^* : \mathit{Hydras} \to \mathit{Worms}$.
- $\mathit{red}(T, n) =$ tree obtained from T by one step of the Hydra Game starting at stage n using the strategy R: *always cut the rightmost head*.
- Then: $\forall n(\mathit{red}(T, n)^* \sqsubset \mathit{next}(T^*, n))$.
- T^* rewrites to $\mathit{red}(T, n)^*$ for all n by the rules of WP.
- Termination of WP implies termination of HG. Hence **PA** $\nvdash$ *EWD*.
- Other approaches (giving more information) based on ordinal notations were given by Lee and Weiermann.
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Gap-embedding

- Friedman's gap-embedding: Let S_{n+1} be the set of all finite sequences of natural numbers smaller than $n+1$. If $s_1 = (a_0, \ldots, a_k)$, $s_2 = (b_0, \ldots, b_m)$ are in S_{n+1} then a strictly monotone function $f : \{0, \ldots, k\} \to \{0, \ldots, m\}$ is a gap-embedding of s_1 into s_2 if the following holds:
  1. $a_i = b_{f(i)}$ for all $i < k$, and
  2. if $f(i) < j < f(i+1)$ then $b_j \geq f(i+1)$ for all $i < k$ and $j < m$.
- We say that s_1 is gap-embeddable in s_2 and write $s_1 \leq_{ge} s_2$.
- Friedman-Simpson: For each n, $\leq_{ge}$ is a wqo of S_{n+1}.
- The just above wqo theorem is unprovable in $\mathbf{ACA}_0$.
- Finite **PA**-independence results follow: long normed finite sequences are wqo under $\leq_{ge}$.

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Worms and gaps

- Sequences of worms are bad with respect to Friedman's gap-embeddability relation.
- The following holds (elementary comb. proof).

$$(s_1 \hookrightarrow_W s_2) \Rightarrow (s_1 \not\leq_{ge} s_2).$$

- The following holds.

$$\forall w(m > n \Rightarrow w_n \not\leq_{ge} w_m).$$

- Long sequences of worms give long bad sequences wrt $\leq_{ge}$. This implies *slowly well-orderedness* **PA**-independence results of Simpson and Schütte.

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Worms and gaps, Beklemishev's results

- (Strong) gap-embeddability is *characterized* by reverse provable implication in GLP^-.
- (Strong gap-embeddability is just gap-embeddability with the extra condition that $\forall j < f(1)(b_j \geq b_{f(1)})$).
- Characterization:

$$w \leq_{sge} w' \Leftrightarrow GLP^- \vdash w' \rightarrow w.$$

- For every n, the extension of GLP^- by n closed modalities corresponding to negations of worms is finitely axiomatizable. (From wqo of $\leq_{sge}$ we get a finite base).
- The just above theorem is unprovable in $\mathbf{ACA}_0$. (It implies that $\leq_{sge}$ is wqo).

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