

# Provability algebras for theories of Tarskian truthpredicates

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- Joint work with Lev Beklemishev.
- Work is still in progress.

## General problem

Extend provability algebra based proof-theoretic analysis to theories stronger than **PA**. Predicative analysis theories are of particular interest.

# Truthpredicates

- Truthpredicates are tightly related to reflection principles and are convenient in our framework.
- Theories of iterated truth are mutually interpretable with various standard theories of predicative strength (ramified analysis, iterated  $\Pi_1^0$ -comprehension).

## Model case: One truthpredicate

- $T(x)$  “ $x$  is the Gödel number of a true arithmetical sentence”.
- Let  $\mathcal{L}(T)$  be the extension of the language of **PA** by  $T$ .

Tarski principles for truth:

- $\forall\phi(\text{At}[\phi] \rightarrow (T[\phi] \leftrightarrow \text{Tr}_0[\phi]));$
- $\forall\phi, \psi(T[\phi \wedge \psi] \leftrightarrow (T[\phi] \wedge T[\psi]));$
- $\forall\phi(T[\neg\phi] \leftrightarrow \neg T[\phi]);$
- $\forall\phi(\cdot)(T[\forall\phi(x)] \leftrightarrow \forall x T\phi[x]).$

Theories:

**BT** = **PRA** + Tarskian truth; **PA(T)** = **BT** + full induction scheme for  $\mathcal{L}(T)$ .

# Reflection principles

Extend arithmetical hierarchy to  $\mathcal{L}(T)$ :

- $\Pi_\omega = \bigcup_{n < \omega} \Pi_n$ ;
- for  $n \geq 1$ ,  $\Sigma_{\omega+n}$  are  $\Sigma_n(T)$ -formulas.

For  $n \geq 1$ , let  $Tr_n$  and  $Tr_{\omega+n}$  be natural truthdefinitions for  $\Sigma_n$  and  $\Sigma_n(T)$  sentences respectively.

For a r. e. extension  $S$  of **BT** in  $\mathcal{L}(T)$ , consider reflection schemes:

- $R_0(S) \equiv \text{Con}(S)$ ;
- $R_n(S) \equiv \forall \sigma \in \Sigma_n(\Box_S \sigma \rightarrow Tr_n(\sigma))$  for  $n \geq 1$ ;

## Reflection principles

- $R_\omega(\mathcal{S}) \Leftrightarrow \{R_n(\mathcal{S}) \mid n < \omega\}$  (uniform reflection);
- $R_{\omega+1}(\mathcal{S}) \Leftrightarrow \forall \phi \in \Pi_\omega(\Box_{\mathcal{S}}(\phi) \rightarrow T(\phi))$  (global arithmetical reflection);
- $R_{\omega+1+n}(\mathcal{S}) \Leftrightarrow \forall \sigma \in \Sigma_n(T)(\Box_{\mathcal{S}}\sigma \rightarrow Tr_{\omega+n}(\sigma))$ , for  $n \geq 1$ ;
- $R_{\omega+\omega}(\mathcal{S}) \Leftrightarrow \{R_\alpha(\mathcal{S}) \mid \alpha < \omega + \omega\}$

Note that  $R_\omega(\mathcal{S})$  and  $R_{\omega+\omega}(\mathcal{S})$  are infinite schemes, not single sentences.

# Induction and Reflection

- **BT** is  $\Pi_\omega$ -conservative over **PRA** (Kotlarski, Krajewski, Lachlan model-theoretically; Halbach syntactically);
- **BT** +  $R_\omega(\mathbf{BT}) \equiv_{\Pi_\omega}$  **PA**;
- **PA(T)** and **ACA** are mutually interpretable;
- **BT** +  $R_{\omega+\omega}(\mathbf{BT}) \equiv \mathbf{PA}(T)$ ;
- **BT** +  $R_{\omega+1}(\mathbf{BT}) \equiv \mathbf{BT} + I\Delta_0(T)$  (Kotlarski model-theoretically).



## Reduction property

Let  $R_\alpha^1(S) = R_\alpha(S)$  and  $R_\alpha^{k+1}(S) = R_\alpha(S + R_n^k(S))$ .

**Th.** Let  $U$  and  $S$  be r. e. theories in  $\mathcal{L}(T)$  and let  $S \vdash U$ . Then provably in **PRA**,

$$U + R_{\alpha+1}(S) \equiv_{\Pi_{1+\alpha}} U + \{R_\alpha^k(S) \mid k < \omega\},$$

provided  $U$  is an extension of **BT** of the following complexity:

- $U \subseteq \Pi_{\alpha+2}$  if  $\alpha < \omega$ ;
- $U \subseteq \Pi_\omega$  if  $\alpha = \omega$ ;
- $U \subseteq \Pi_{\alpha+1}$  if  $\alpha > \omega$ .

# Reduction property

## Proof

- For  $\alpha < \omega$ , the standard reduction by Beklemishev;
- For  $\alpha > \omega$ , a relativization of the standard reduction:  
 $\Pi_k \mapsto \Pi_k(\mathcal{T})$ ;
- Let  $\alpha = \omega$ . For  $S = \mathbf{BT}$  this result is due to H. Kotlarski (model-theoretical proof). **PRA**-formalization is new:
  - we formalize an analogue of Kotlarski's argument in **WKL**<sub>0</sub> and use  $\Pi_2$ -conservativity of **WKL**<sub>0</sub> over **PRA** (Friedman).

## Reflection calculus $\mathbf{RC}_\Omega$

Fix some ordinal  $\Omega$  or let  $\Omega = \mathbf{On}$ . For  $\mathcal{L}(T)$ , putting  $\Omega = \omega^2 + 1$  is enough.

- Language:  $A ::= \top \mid A_1 \wedge A_2 \mid \alpha A, \quad \alpha < \Omega$ .
- Derive sequents  $A \vdash B$ .  $A \sim B$  stands for both  $A \vdash B$  and  $B \vdash A$ .
- Do not write  $\top$  for brevity's sake.

## $\mathbf{RC}_\Omega$ : definition

Rules of  $\mathbf{RC}_\Omega$ :

- 1  $A \vdash A$ ;  $A \vdash \top$ ; if  $A \vdash B$  and  $B \vdash C$  then  $A \vdash C$ ;
- 2  $A \wedge B \vdash A, B$ ; if  $A \vdash B$  and  $A \vdash C$  then  $A \vdash B \wedge C$ ;
- 3  $\alpha \alpha A \vdash \alpha A$ ; if  $A \vdash B$  then  $\alpha A \vdash \alpha B$ ;
- 4  $\alpha A \vdash \beta A$  for  $\alpha > \beta$ ;
- 5  $\alpha A \wedge \beta B \vdash \alpha(A \wedge \beta B)$  for  $\alpha > \beta$  and  $\beta \notin \text{Lim}$ .
- 6  $(\alpha + 1)\alpha A \sim (\alpha + 1)A$  for  $\alpha \in \text{Lim}$ .

Unlike  $\mathbf{GLP}_\Omega$ , here we have, e. g.,

$$(\omega + 1) \wedge \omega(\omega + 1) \not\sim (\omega + 1)\omega(\omega + 1) \sim (\omega + 1)(\omega + 1).$$

## $\mathbf{RC}_\Omega$ : arithmetical interpretation

Let  $S$  be a r. e. extension of  $\mathbf{BT}$  in  $\mathcal{L}(T)$ . Interpret formula  $A$  by scheme  $A_S$ :

- $\top_S = \emptyset$ ;  $(A \wedge B)_S = (A_S \cup B_S)$ ;
- $(\alpha A)_S = R_\alpha(S + A_S)$ .

**Th.** If  $A \vdash B$  in  $\mathbf{RC}_\Omega$ , then  $S + A_S \vdash B_S$ .

Note that  $\mathbf{RC}_\Omega$  is not arithmetically complete. E. g. the correct principle

$$0(\omega(\omega + 1) \wedge \omega(\omega + 2)) \vdash 0(\omega(\omega + 1) \wedge (\omega + 2))$$

is not derivable in  $\mathbf{RC}_\Omega$ .

## $\mathbf{RC}_\Omega$ : an ordinal notation system

- Denote  $A <_\alpha B \Leftrightarrow B \vdash \alpha A$ .
- Let  $W$  denote the set of all  $\mathbf{RC}_\Omega$  words (i. e. formulas without  $\wedge$ ).
- Denote by  $W_\alpha$  the set of words in the alphabet  $\{\beta \mid \alpha \leq \beta\}$ .

**Th.**  $(W/\sim, <_0)$  is a well-ordering.

## Words and ordinals

0	...	1	...	101	...	11	...	$n$	...	$\omega$
1	...	$\omega$	...	$\omega 2$	...	$\omega^2$	...	$\omega n$	...	$\epsilon_0$
$0\omega$	...	$10\omega$	...	$\omega 0\omega$	...	$1\omega$	...	$\omega\omega$	...	$(\omega + 1)$
$\epsilon_0 + 1$	...	$\epsilon_0 + \omega$	...	$\epsilon_0 \cdot 2$	...	$\epsilon_0 \cdot \omega$	...	$\epsilon_1$	...	$\epsilon_\omega$
$\omega(\omega + 1)$	...	$(\omega + 1)(\omega + 1)$	...							
$\epsilon_{\omega+1}$	...	$\epsilon_{\omega \cdot 2}$	...							

**Th.** If  $\omega^\alpha < \Omega$ , then  $o(W_{\omega^\alpha}) = \{0\} \cup C$ , where  $C$  is an initial segment of  $Cr_\alpha$ . In particular, if  $\Omega = \text{On}$ , then  $C = Cr_\alpha$ .

## Iterated reflection

- $o_\alpha(A)$  is the order type of  $\{B \in W_\alpha \mid B <_\alpha A\}$ ;
- let  $S_\beta^\alpha \equiv S + \{R_\alpha(S_\gamma^\alpha) \mid \gamma < \beta\}$ .

**Th.** Let  $S$  be a r. e.  $\Pi_{1+\alpha}$ -extension of **BT**. Provably in **PRA**, for any  $A \in W_\alpha$ ,  $S + A_S \equiv_{\Pi_{1+\alpha}} S_{o_\alpha(A)}^\alpha$ .

**Cor.** For  $n < \omega$ ,

- $I\Delta_0(T) \equiv \mathbf{BT} + R_{\omega+1}(\mathbf{BT}) \equiv_{\Pi_\omega} \mathbf{BT}_\omega^\omega \equiv_{\Pi_{1+n}} \mathbf{BT}_{\epsilon_\omega}^n$ ;
- $\mathbf{PA}(T) \equiv \mathbf{BT} + R_{\omega+\omega}(\mathbf{BT}) \equiv_{\Pi_\omega} \mathbf{BT}_{\epsilon_0}^\omega \equiv_{\Pi_{1+n}} \mathbf{BT}_{\epsilon_{\epsilon_0}}^n$ .



# Reflection and conservativity calculus

## **RCC**<sub>Ω</sub>

- **RC**<sub>Ω</sub> is not arithmetically complete.
- A candidate for complete system is **RCC**<sub>Ω</sub>:
  - Language of **RC**<sub>Ω</sub>.
  - Derive sequents  $A \vdash B$   
and for  $\alpha < \Omega$ ,  $A \vdash_{\alpha} B$       “ $B$  is  $\Pi_{1+\alpha}$ -conservative over  $A$ ”.

## RCC<sub>Ω</sub>: definition

RCC<sub>Ω</sub> rules:

- ① Rules of RC<sub>Ω</sub>;
- ② if  $A \vdash_{\alpha} B$  and  $B \vdash_{\alpha} C$  then  $A \vdash_{\alpha} C$ ; if  $A \vdash_{\alpha} B$  then  $\alpha A \vdash \alpha B$ ;
- ③ if  $A \vdash B$  then  $A \vdash_{\alpha} B$ ; if  $A \vdash_{\alpha} B$  then  $A \vdash_{\beta} B$  for  $\beta < \alpha$ ;
- ④  $\alpha A \vdash_{\alpha} A$ ; if  $A \vdash_{\alpha} \alpha B$  then  $A \vdash \alpha B$ ;
- ⑤  $A \vdash_{\alpha} B$  then  $\beta C \wedge A \vdash_{\alpha} \beta C \wedge B$  for  $\beta < \alpha$ ;  $A \vdash_{\lambda} B$  then  $\lambda C \wedge A \vdash_{\lambda} \lambda C \wedge B$

## $\text{RCC}_\Omega$ : some known facts

- $\text{RCC}_\Omega$  is correct w.r.t. the intended arithmetical interpretation;
- for any formula  $A$ ,  $\alpha A$  is  $\text{RCC}_\Omega$ -equivalent to some word  $B$ ;
- the set of  $\text{RCC}_\Omega$ -derivable equivalences  $A \sim B$  is arithmetically complete.