

Models of provability logic

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First International Wormshop
17 April 2012

Gödel-Löb logic

Language:

$$p \quad \neg\varphi \quad \varphi \wedge \psi \quad \Box\varphi$$

Axioms:

- ▶ $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- ▶ $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$ (Löb's axiom)

Second incompleteness theorem:

$$\Box\Diamond T \rightarrow \Box\perp$$

Arithmetic interpretation

An **arithmetic interpretation** assigns a formula p^* in the language of arithmetic to each propositional variable p .

- ▶ $p \mapsto p^*$
- ▶ $\square \mapsto Prv_{PA}$

Theorem (Solovay)

If $GL \not\vdash \varphi$ then for some arithmetic interpretation $$, $PA \not\vdash \varphi^*$.*

Relational semantics

Kripke models:

- ▶ Frames: Well-founded partial orders $\langle W, < \rangle$
- ▶ Valuations: $\llbracket \varphi \rrbracket \subseteq \mathcal{P}(W)$,

$$w \in \llbracket \Box \varphi \rrbracket \Leftrightarrow \forall v < w, v \in \llbracket \varphi \rrbracket$$

Theorem

GL is sound for $\langle W, < \rangle$ if and only if $<$ is well-founded.

Further, GL is complete for the class of well-founded frames and enjoys the finite model property.

Topological semantics:

- ▶ GL-spaces: **scattered** topological spaces $\langle X, \mathcal{T} \rangle$
Scattered: Every non-empty subset contains an isolated point.
- ▶ Valuations: dA is the set of **limit points** of A .

$$\llbracket \diamond \varphi \rrbracket = d \llbracket \varphi \rrbracket .$$

GL is also **sound and complete** for this interpretation.

Some scattered spaces

- ▶ A finite partial order $\langle W, \leq \rangle$ with the **downset topology**
- ▶ An ordinal ξ with the **initial segment topology**
- ▶ An ordinal ξ with the **order topology**

Non-scattered:

- ▶ The real line
- ▶ The rational numbers
- ▶ The Cantor set

Polymodal Gödel-Löb

GLP: Contains one modality $[n]$ for each $n < \omega$.

Axioms:

$$\begin{array}{ll} [n](\varphi \rightarrow \psi) \rightarrow ([n]\varphi \rightarrow [n]\psi) & (n < \omega) \\ [n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi & (n < \omega) \\ [n]\varphi \rightarrow [m]\varphi & (n < m < \omega) \\ \langle n \rangle \varphi \rightarrow [m]\langle n \rangle \varphi & (n < m < \omega) \end{array}$$

(Possible) arithmetic interpretation:

$[n]\varphi \equiv$ “ φ is provable using n instances of the ω -rule”.

Introduced by Japaridze in 1988.

Kripke semantics

Frames:

$$\langle W, \langle \prec_n \rangle_{n < \omega} \rangle$$

$$[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi:$$

Valid iff \prec_n is well-founded

$$[n]\varphi \rightarrow [n+1]\varphi:$$

Valid iff $w \prec_{n+1} v \Rightarrow w \prec_n v$

$$\langle n \rangle \varphi \rightarrow [n+1] \langle n \rangle \varphi:$$

Valid iff

$$v \prec_n w \text{ and } u \prec_{n+1} w \Rightarrow v \prec_n u$$

Even GLP_2 has **no non-trivial Kripke models**.

Topological semantics

Spaces:

$$\langle X, \langle \mathcal{T}_n \rangle_{n < \omega} \rangle$$

Write d_n for the limit point operator on \mathcal{T}_n .

$[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi$: Valid iff \mathcal{T}_n is scattered

$[n]\varphi \rightarrow [n+1]\varphi$: Valid iff $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$

$\langle n \rangle \varphi \rightarrow [n+1]\langle n \rangle \varphi$: Valid iff

$$A \subseteq X \Rightarrow d_n A \in \mathcal{T}_{n+1}$$

Topological completeness

Beklemishev, Gabelaia: GLP is complete for the class of GLP-spaces

The proof uses **non-constructive** methods.

Blass: It is consistent with ZFC that the **canonical ordinal spaces** for GLP_2 are all trivial

Beklemishev: It is also consistent with ZFC that GLP_2 is complete for its canonical ordinal spaces

Bagaria More generally, **for all n** it is consistent with ZFC that GLP_n has non-trivial canonical ordinal spaces but GLP_{n+1} does not.

The closed fragment

Written GLP^0 , it does not allow propositional variables (only \perp).

Beklemishev: GLP^0_ω may be used to perform ordinal analysis of PA, its natural subtheories and some extensions.

Theorem (Ignatiev)

There is a Kripke frame \mathfrak{J} such that GLP^0_ω is sound and complete for \mathfrak{J} .

Ignatiev's model of GLP^0

Given an ordinal $\xi = \alpha + \omega^\beta$, define $l\xi = \beta$ ($l0 = 0$).

Ignatiev's model:

$$\mathfrak{J} = \langle D, \langle <_n \rangle_{n < \omega} \rangle$$

- ▶ $D = \{f : \omega \rightarrow \varepsilon_0 : \forall n f(n+1) \leq lf(n)\}$
- ▶ $f <_n g$ if $f(m) = g(m)$ for $m < n$ and $f(n) < g(n)$

Example:

$$\langle \omega^{\omega+1}, \omega, 0, \dots \rangle <_2 \langle \omega^{\omega+1}, \omega, 1, 0, \dots \rangle$$

Frame conditions

Ignatiev's model **does not satisfy all frame conditions.**

$[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi$:

$<_n$ is based on an ordinal and hence well-founded

$[n]\varphi \rightarrow [n+1]\varphi$:

$$\langle \omega, 1, 0, \dots \rangle \not<_0^1 \langle \omega, 0, 0, \dots \rangle$$

$\langle n \rangle \varphi \rightarrow [n+1]\langle n \rangle \varphi$:

$$\begin{array}{r} \langle \omega^\omega, 0, 0, 0, \dots \rangle <_1 \langle \omega^\omega, \omega, 1, 0, \dots \rangle \\ \langle \omega^\omega, \omega, 0, 0, \dots \rangle <_2 \langle \omega^\omega, \omega, 1, 0, \dots \rangle \\ \hline \langle \omega^\omega, 0, 0, 0, \dots \rangle <_1 \langle \omega^\omega, \omega, 0, 0, \dots \rangle \end{array}$$

The main axis

Definition

A sequence $f : \omega \rightarrow \varepsilon_0$ is **exact** if for all n ,

$$f(n+1) = \ell f(n).$$

Main axis: Set of exact sequences.

Lemma

Every closed formula which is satisfied on \mathfrak{J} is satisfied on the main axis.

Is GLP^0 sound for \mathfrak{J} ?

Icard topologies

Icard defined a structure

$$\mathfrak{T} = \langle \varepsilon_0, \langle \mathcal{T}_n \rangle_{n < \omega} \rangle.$$

Generalized intervals:

$$(\alpha, \beta)_n = \{\vartheta : \alpha < \ell^n \vartheta < \beta\}.$$

\mathcal{T}_n is generated by intervals of the form

- ▶ $(\alpha, \beta)_m$ for $m < n$
- ▶ $[0, \beta)_m$ for $m \leq n$

Topological conditions

Icard's model does not satisfy all frame conditions either.

$[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi$:

\mathcal{T}_n is scattered since \mathcal{T}_0 is.

$[n]\varphi \rightarrow [n+1]\varphi$: \mathcal{T}_{n+1} is always a refinement of \mathcal{T}_n .

$\langle n \rangle \varphi \rightarrow \langle n+1 \rangle \varphi$: The point

$$\omega^\omega = \lim_{n \rightarrow \omega} \omega^n$$

should be isolated in \mathcal{T}_2 .

Ignatiev vs. Icard

Define $\vec{l}: \varepsilon_0 \rightarrow D$ by

$$\vec{l}\xi = \langle \xi, l\xi, l^2\xi, \dots, l^n\xi, \dots \rangle$$

Lemma

For every $\xi < \varepsilon_0$,

$$\langle \mathcal{T}, \xi \rangle \models \varphi \Leftrightarrow \langle \mathcal{I}, \vec{l}\xi \rangle \models \varphi$$

Corollary

\mathcal{I} and \mathcal{T} satisfy the same set of formulas.

Soundness and completeness

Theorem

GLP^0 is sound for both \mathfrak{J} and \mathfrak{T} .

Proof.

$[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi$: Valid on both \mathfrak{J} and \mathfrak{T} .

$[n]\varphi \rightarrow [n+1]\varphi$: Valid on \mathfrak{T} .

$\langle n \rangle \varphi \rightarrow [n+1]\langle n \rangle \varphi$: Valid on \mathfrak{J} . □

Theorem (Ignatiev, Icard)

GLP^0 is complete for both \mathfrak{J} and \mathfrak{T} .

Concluding remarks

- ▶ GL is very nice as a modal logic, but only takes us so far
- ▶ GLP is very useful! (Ordinal analysis, ordinal notation systems, unprovable statements...)
- ▶ **But** GLP is tougher to work with
 - ▶ No Kripke frames
 - ▶ Topological completeness is **hard**
- ▶ The closed fragment gives us a good **middle ground**
- ▶ Here we have Kripke models, **simple** topological models.
- ▶ **Future work:** Generalize to **stronger theories**
- ▶ **Tomorrow's talk:** Modalities beyond ω ??

Thank you!