Predicativity – Part II

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Predicative reducibility

New interest in predicativity: Importance of predicativity - or better: predicatively reducible systems - for mathematical practice.

Definition

A formal theory is called *predicatively reducible* iff every arithmetic sentence provable in T is also provable in $\mathsf{AUT}(\Pi^0_1)$, i.e.

$$T \ \leq_{\Pi^0_\infty} \ \mathsf{AUT}(\Pi^0_1).$$

Example

Systems like $(\Sigma^1_1\text{-DC}) + (\mathsf{BR})$ are predicatively reducible but not predicative in the strong sense since

$$L_{\Gamma_0} \cap \mathrm{Pow}(\mathbb{N}) \not\models (\Sigma^1_1\text{-DC}).$$

"Modern" predicatively reducible systems

Some characteristics

- Comparatively strong set-existence axioms.
- Fairly weak induction principles; for example, systems are often not closed under the Bar Rule (BR).

Friedman's ATR_0

- The schema of arithmetic transfinite recursion: for every arithmetic formula $A[X, y]$,

$$\forall R(\mathrm{WO}[R] \rightarrow \forall X \exists Y \, \mathrm{Hier}_A[R, X, Y]). \qquad \text{(ATR)}$$

- $\mathsf{ATR}_0 \; := \; \mathsf{ACA}_0 + \text{(ATR)}$.

“Modern” predicatively reducible systems (cont.)

The fixed point theory FP_0

- The fixed point axiom: For every arithmetic formula $A[X^+, y]$ with only positive occurrences of X (but possibly further set parameters),

$$\exists X \forall n (n \in X \leftrightarrow A[X, n]). \tag{FP}$$

- $\mathsf{FP}_0 := \mathsf{ACA}_0 + \text{(FP)}$.

Theorem (Avigad)

ATR_0 *and* FP_0 *prove the same formulas of second order arithmetic.*

"Modern" predicatively reducible systems (cont.)

Some general remarks

1. ATR_0 is one of the five main systems of reverse mathematics.
2. It is easy to show that ATR_0 has at least the proof-theoretic strength of $\mathsf{AUT}(\Pi^0_1)$. For the upper bound see below.
3. FP_0 provides a natural framework for most (all?) extensions of Kripke-Feferman truth theories; partial truth definitions are treated as fixed points of suitable operator forms.
4. Martin-Löf's theory $\mathsf{ML}(U_1, U_2, \ldots)$ with universes can be modeled in FP_0; proof of Hankocks conjecture.

Iterated admissible sets without foundation

The theory KPu

- A theory of sets above the natural numbers as urelemente.
- Every object is a natural number or a set, but not both; the natural numbers form a set.
- As axioms for the natural numbers we have the axioms of PA.
- Set-theoretic axioms: extensionality, pairing, transitive hull, Δ_0 separation, Δ_0 collection.
- Complete induction on the naturals and $\in$ induction for all formulas.

Theorem (Jä)

$$\mathsf{KPu} \equiv \mathsf{ID}_1.$$

Iterated admissible sets without foundation (cont.)

The theories KPu^0 and KPi^0

- $\mathsf{KPu}^0 := \begin{cases} \text{KPu with complete induction on the natural numbers} \\ \text{restricted to sets and no } \in \text{ induction.} \end{cases}$
- Now add a new unary relation symbol Ad and the axiom

$$\forall x(\mathrm{Ad}(x) \rightarrow \mathrm{N} \in x \wedge \text{transitive } x \wedge x \models \mathsf{KPu}^0).$$

- $\mathsf{KPi}^0 := \mathsf{KPu}^0 + \forall x \exists y(x \in y \wedge \mathrm{Ad}(y)).$

Theorem (Jä)

$$\mathsf{KPi}^0 \equiv \mathsf{AUT}(\Pi^0_1).$$

Basic idea of unfolding theories

Main question

Given a schematic system S, which operations and predicates, and which principles concerning them, ought to be accepted if one has accepted S?

In particular, what is inherent in the structure of the natural numbers and the schema of complete induction for arbitrary properties?

Underlying the idea of unfolding for a specific S are:

- General notions of *operation* and *predicate* belonging to a universe encompassing the universe of discourse of S.
- Operations are not bound to any specific mathematical domain; they are pre-mathematical. Operations apply to operations and predicates.
- Operations form a partial combinatory algebra with pairing and a possibly additional properties.

Basic idea of unfolding theories (cont.)

- Predicates are equipped with a membership relation to express that given elements satisfy a predicate's defining property.
- The universe of discourse of S is associated with an unary relation symbol U_S, and all axioms of S are relativized to U_S.
- For every function symbol f of S there exists a corresponding element f^* in the combinatory part; every relation symbol R of S is represented by predicate R^*.
- The logical operations of S determine corresponding operations on predicates.
- The free predicate variables of S give raise to a rule of substitution.

Unfolding non-finitist arithmetic NFA

The theory NFA

- Formulated in a usual first order language with a constant 0, unary function symbols Sc (successor) and Pd (predecessor), and free relation predicate variables $P, Q, R, \ldots$ of all finite arities.
- Axioms of NFA:
  - $\mathrm{Sc}(x) \neq x \wedge \mathrm{Pd}(\mathrm{Sc}(x)) = x$,
  - $P(0) \wedge \forall x(P(x) \to P(\mathrm{Sc}(x))) \to \forall x P(x)$.
- Rule of substitution for all formulas A and B:

$$\frac{A[P]}{A[B]}\,.$$

Operational unfolding $\mathcal{U}_0(\mathsf{NFA})$ of NFA

The language $\mathcal{L}_0$ of $\mathcal{U}_0(\mathsf{NFA})$

- Extension of the language of NFA.
- Constants for operations on individuals, namely $\mathbf{sc}, \mathbf{pd}$ (successor, predecessor), $\mathbf{k}$, $\mathbf{s}$ (combinators), $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1$ (pairing, unpairing), $\bot, \top$ (truth values), $\mathbf{d}$ (definition by cases), and $\mathbf{e}$ (equality).
- Binary function symbol $\cdot$ for partial term application.
- Unary relation symbols $\downarrow$ (defined) and N (natural numbers).

Terms $(r, s, t, \ldots)$ of $\mathcal{L}_0$

$$r, s, t \quad ::= \quad \text{variables} \mid \text{constants} \mid \mathrm{Sc}(t) \mid \mathrm{Pd}(t) \mid (s \cdot t).$$

Operational unfolding $\mathcal{U}_0$(NFA) of NFA (cont.)

Abbreviations

$$
\begin{aligned}
(st),\ st,\ s(t) \quad &\text{for} \quad (s \cdot t), \\
t \in \mathrm{N} \quad &\text{for} \quad \mathrm{N}(t), \\
s \simeq t \quad &\text{for} \quad (s{\downarrow} \vee t{\downarrow}) \rightarrow (s = t).
\end{aligned}
$$

Formulas $(A, B, C, \ldots)$ of $\mathcal{L}_0$

As usual from the terms of $\mathcal{L}_0$.

Logic of $\mathcal{U}_0$(NFA)

Classical logic of partial terms due to Beeson.

Operational unfolding $\mathcal{U}_0(\mathrm{NFA})$ of NFA (cont.)

Axioms of $\mathcal{U}_0(\mathrm{NFA})$

1. NFA axioms
   - $0 \in \mathrm{N} \;\wedge\; (\forall x \in \mathrm{N})(\mathrm{Sc}(x) \in \mathrm{N} \wedge \mathrm{Pd}(x) \in \mathrm{N})$.
   - $a \in \mathrm{N} \;\rightarrow\; (\mathrm{Sc}(a) \neq 0 \wedge \mathrm{Pd}(\mathrm{Sc}(a)) = a)$.
   - $P(0) \wedge (\forall x \in \mathrm{N})(P(x) \rightarrow P(\mathrm{Sc}(x))) \;\rightarrow\; (\forall x \in \mathrm{N})P(x)$.
2. Partial combinatory algebra, pairing, definition by cases
   - $\mathbf{k}ab = a \;\wedge\; \mathbf{s}ab\!\downarrow \;\wedge\; \mathbf{s}abc \simeq (ac)(bc)$.
   - $\mathbf{p}_0(\mathbf{p}ab) = a \;\wedge\; \mathbf{p}_1(\mathbf{p}ab) = b \;\wedge\; \mathbf{d}ab\top = a \;\wedge\; \mathbf{d}ab\bot = b$.
   - $P(0) \wedge (\forall x \in \mathrm{N})(P(x) \rightarrow P(\mathrm{Sc}(x))) \;\rightarrow\; (\forall x \in \mathrm{N})P(x)$.
3. Equality on the natural numbers
   - $(\forall x, y \in \mathrm{N})(\mathbf{e}xy = \top \vee \mathbf{e}xy = \bot)$.
   - $(\forall x, y \in \mathrm{N})(\mathbf{e}xy = \top \;\leftrightarrow\; x = y)$.

Operational unfolding $\mathcal{U}_0(\mathsf{NFA})$ of NFA (cont.)

Rule of substitution of $\mathcal{U}_0(\mathsf{NFA})$

For all formulas formulas A and B of $\mathcal{L}_0$:

$$\frac{A[P]}{A[B]} \,.$$

Theorem (Feferman, Strahm)

$$\mathcal{U}_0(\mathsf{NFA}) \;\equiv\; \mathsf{PA}.$$

Full predicate unfolding $\mathcal{U}(\mathsf{NFA})$ of NFA

The language $\mathcal{L}$ of $\mathcal{U}(\mathsf{NFA})$

- Extension of the language $\mathcal{L}_0$ of $\mathcal{U}_0(\mathsf{NFA})$.
- Additional constants **nat** (natural numbers), **eq** (equality), $\mathbf{pr}_P$ (free predicate P), **inv** (inverse image), **neg** (negation), **conj** (conjunction), **un** (universal quantification), and **join** (disjoint union).
- A new unary relation symbol Π for codes of predicates and a binary relation symbol $\in$ for expressing elementhood between individuals and (codes of) predicates.

Terms $(r, s, t, \dots)$ and formulas $(A, B, C, \dots)$ of $\mathcal{L}$

Based on the vocabulary of $\mathcal{L}$ as above.

Full predicate unfolding $\mathcal{U}$(NFA) of NFA (cont.)

Axioms of $\mathcal{U}$(NFA)

1. All axioms of $\mathcal{U}_0$(NFA).
2. Predicate axioms
   - $\Pi(\mathbf{nat}) \ \wedge \ \forall x(x \in \mathbf{nat} \ \leftrightarrow \ \mathrm{N}(x))$.
   - $\Pi(\mathbf{eq}) \ \wedge \ \forall x(x \in \mathbf{eq} \ \leftrightarrow \ \exists y(x = \mathbf{p}yy))$.
   - $\Pi(\mathbf{p}_P) \ \wedge \ \forall \vec{x}((\vec{x} \in \mathbf{p}_P \ \leftrightarrow \ P(\vec{x}))$.
   - $\Pi(a) \ \rightarrow \ \Pi(\mathbf{inv}(a, f)) \wedge \forall x(x \in \mathbf{inv}(a, f) \leftrightarrow fx \in a)$.
   - $\Pi(a) \ \rightarrow \ \Pi(\mathbf{neg}(a)) \wedge \forall x(x \in \mathbf{neg}(a) \leftrightarrow x \notin a)$.
   - $\Pi(a) \wedge \Pi(b) \ \rightarrow \ \Pi(\mathbf{conj}(a, b)) \wedge \forall x(x \in \mathbf{conj}(a, b) \leftrightarrow x \in a \wedge x \in b)$.
   - $\Pi(a) \ \rightarrow \ \Pi(\mathbf{un}(a)) \wedge \forall x(x \in \mathbf{un}(a) \leftrightarrow (\forall y \in \mathrm{N})(\mathbf{p}xy \in a))$.
   - $(\forall x \in \mathrm{N})\Pi(fx) \ \rightarrow \ \Pi(\mathbf{join}(f)) \wedge \forall x(x \in \mathbf{join}(f) \leftrightarrow J[f, x])$,

     where $\quad J[f, u] \ :\Leftrightarrow \ (\exists y \in \mathrm{N})\exists z(u = \mathbf{p}yz \wedge z \in fy)$.

Full predicate unfolding $\mathcal{U}(\mathsf{NFA})$ of NFA (cont.)

Rule of substitution of $\mathcal{U}(\mathsf{NFA})$

For all formulas formulas A of the language $\mathcal{L}_0$ and all formulas B of $\mathcal{L}$:

$$\frac{A[P]}{A[B]}\ .$$

Theorem (Feferman, Strahm)

$$\mathcal{U}(\mathsf{NFA}) \ \equiv \ \mathsf{AUT}(\Pi^0_1).$$

Beyond predicativity

Adding the schema (IS) of full induction to, e.g., ATR_0 and FP_0

- $\mathsf{ATR} = \mathsf{ATR}_0 + (\mathrm{IS})$ and $\mathsf{FP} := \mathsf{FP}_0 + (\mathrm{IS})$.
- $|\mathsf{ATR}| = |\mathsf{FP}| = \Gamma_{\varepsilon_0}$, hence ATR and FP are not predicatively reducible.
- However, methods used in the proof-theoretic analysis of ATR and FP are those used for the analysis of ATR_0 and FP_0.
- No methodological difference between ATR_0 and ATR and between FP_0 and FP.
- This is in strong contrast to the standard proof-theoretic treatment of, for example, ID_1.

Predicative – metapredicative – impredicative

Metapredicative systems

- Systems which are not predicatively reducible, i.e. which have a proof-theoretic ordinal greater than Γ_0.
- But whose proof-theoretical analysis can be carried through without making use of any impredicative methods (such as collapsing techniques).

Examples

1. $\mathsf{KPm}^0 := \mathsf{KPi}^0 + (\Pi_2\text{-Ref})^{\mathrm{Ad}}$, $|\mathsf{KPm}^0| = \varphi\omega 00 > \varphi 100 = \Gamma_0$.
2. $\mathsf{KPi}^0 + (\Pi_n\text{-Ref})$.
3. $\mathsf{ACA}_0 + (\Pi^1_n\text{-BI})$.

Predicative – metapredicative – impredicative (cont.)

Impredicative systems

- The finite proofs of a given theory T are displayed as (recursively) uncountable proofs of a suitable semiformal system.
- For these proofs partial cut elimination is carried through and – in order to do so – typically an interplay between collapsing of uncountable to countable trees and boundedness techniques is required.
- The cut-free countable proofs yield proof-theoretic bounds.

Questions

1. Can the informal notion of metapredicativity been made precise?
2. What is the range of metapredicativity?