Tutorial on provability algebras
Lecture 2
Manipulating and ordering worms

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- Turing progressions can be used for an ordinal analysis:
- "how often should I iterate a finitistic base theory T as to approximate a target theory U: $T_\xi \approx U$"

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- We often not distinguish a modal formula and its interpretation

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- **Proposition:** $T + \langle n+1 \rangle_T \top$ is a Π_{n+1} conservative extension of $T + \{\langle n \rangle_T^k \top \mid k \in \omega\}$.

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Similarly, we inductively define for each ordinal α the set of worms S_α where all ordinals are at least α as $\top \in S_\alpha$ and $A \in S_\alpha \wedge \beta \geq \alpha \Rightarrow \langle\beta\rangle A \in S$.

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- Various axioms can be restricted to worms

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- To get generalizations of this lemma beyond ϵ_0 one should consider more than ω modalities.
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- This talk focuses on the modal calculus involved in such generalizations.

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3. *If* $A \in S_{\alpha+1}$*, then* $\mathsf{GLP} \vdash A \wedge \langle\alpha\rangle B \leftrightarrow A\alpha B$*;*

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- We will order the worms based on these sort of implications

Definition ($<, <_\alpha, o, o_\alpha$)

We define a relation $<_\alpha$ on $S_\alpha \times S_\alpha$ by

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Instead of $<_0$ and o_0 we shall write $<$ and o, respectively.

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- If irreflexive, $<_\alpha$ is a well-order on $S_\alpha \times S_\alpha$ ([B])

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Proof.

This is immediate if we conceive α^n as $\lambda\alpha\lambda \ldots \lambda\alpha\lambda$. □

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$$\textit{If} \quad \mathrm{GLP} \vdash A_1 \to \langle \alpha + 1 \rangle A_0, \ \textit{then}\ \mathrm{GLP} \vdash A \leftrightarrow A_1 \text{rest}.$$

Proof

$$\begin{array}{rcl} \mathrm{GLP} \vdash \quad A & \to & A_1 \wedge \alpha A_0 \text{rest} \\ & \to & A_1 \wedge \text{rest} \\ & \to & A_1 \text{rest}. \end{array}$$

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Proof.

Apply (inductively) the lemma to ‘subworms in violating order’ □

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Lemma

The map $o : (\mathrm{BNF}, <_0) \to (\mathrm{Ord}, <)$ *defines an isomorphism.*

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We need an operation that lowers ordinals in worms to have our calculus going

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- $\forall\,\zeta{<}\xi\,\exists!\eta\;\zeta+\eta=\xi$
  (We will denote this unique η by $-\zeta+\xi$),

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Likewise, if $A \in S_\alpha$ we denote by $\alpha \downarrow A$ the worm that is obtained by replacing simultaneously each β in A by $-\alpha + \beta$.

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For $A \in S_\alpha$ we have

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- **definition**[e^{α}] We define e^{α} to be the function that enumerates $o(S_{\alpha})$.

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Proof.

$A <_\alpha B \;\Leftrightarrow\; A <_0 B \;\Leftrightarrow\; o(A) < o(B)$ for $A, B \in S_\alpha$.

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By composing the isomorphisms

$$o \circ \alpha\!\uparrow \circ\, o^{-1} : (\mathrm{Ord}, <) \cong (o(S_\alpha), <).$$

The following diagram commutes.

$$\begin{array}{ccc} \mathsf{Wrm} & \xrightarrow{\alpha\uparrow} & \mathsf{Wrm} \\ \downarrow{\scriptstyle o} & & \downarrow{\scriptstyle o} \\ \mathsf{Ord} & \xrightarrow{e^{\alpha}} & \mathsf{Ord} \end{array}$$

So o behaves like a functor between the category of worms and the category of ordinals.

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We may assume $A \in$BNF, so
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Thus, knowing e^{α} completes our calculus

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- The closed fragment of GLP provides a “natural” ordinal notation system

END OF TUTORIAL 2

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3. If $\log^*(\alpha) > 0$, then $\mathsf{w}(\alpha) = \log^*(\alpha) \uparrow \mathsf{w}(l^{\log^*(\alpha)}(\alpha))$.