Decidability of the elementary theory of the free **0**-generated **GLP**-algebra and some related questions.

Fedor Pakhomov

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GLP$_n$-algebra is a structure $\mathfrak{B}$ of the signature
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$\mathbf{GLP}_1$-algebras are known as Magari algebras.

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Semilattices

$$\langle k \rangle x \leftrightharpoons \sim[k]\sim x$$

$W \subset \mathfrak{G}$ is the set generated from $\mathbf{1}$ by $\langle 0 \rangle, \langle 1 \rangle, \ldots$
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$\mathrm{Th}(W_n, \wedge)$ *is decidable iff $n \leq 2$.*

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Theorem (J.R. Büchi '60)

Suppose α is an ordinal. Then $\mathrm{Th}(\alpha, <)$ is decidable. Practically weak monadic theory of $(\alpha, <)$ is decidable

Ordinal notations up to ε_0

Some ordinals:

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- $\mathrm{Th}(W_3, <_0, \langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle)$ *is decidable.*

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Some Functors and Morphisms

$\mathbf{O}_n^m \colon \mathcal{M}_n \hookrightarrow \mathcal{M}_m$ is the forgetful functor ($m \leq n$).

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Lemma

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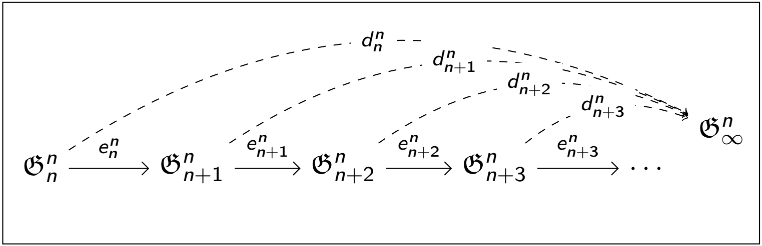
$\mathcal{M}_n$:



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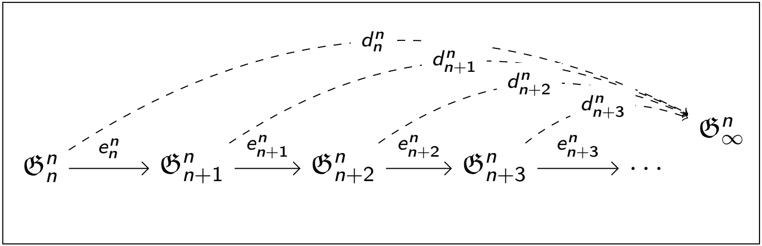
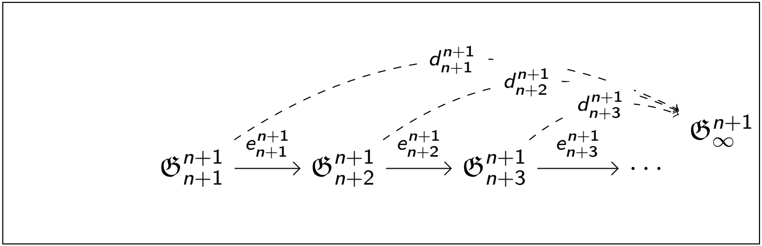


$\mathcal{M}_{n+1}$:



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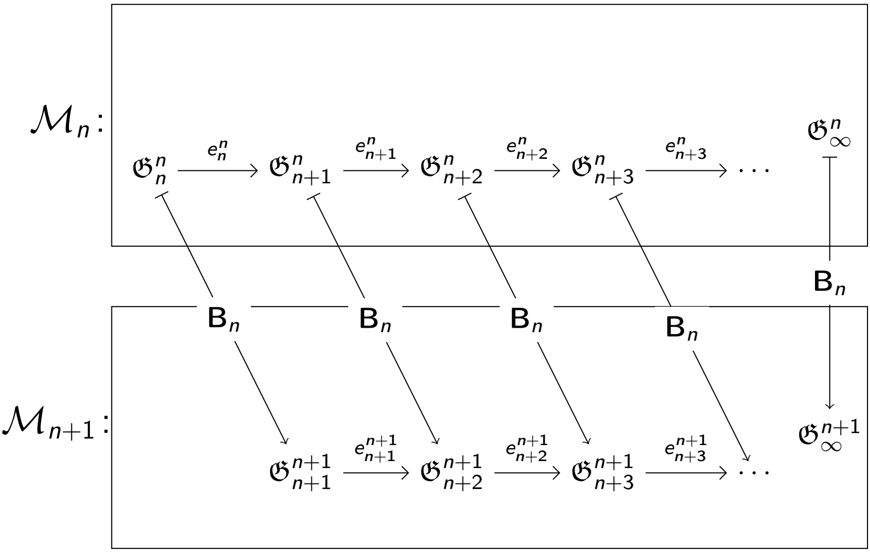

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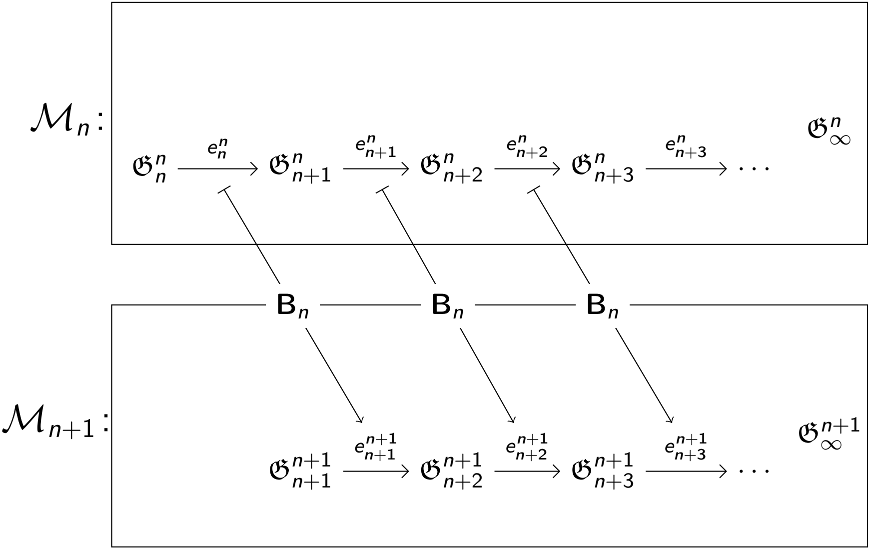

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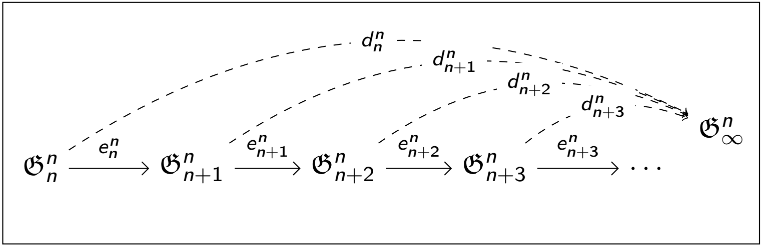

We have $\mathbf{B}_n(d^n_{n+i}) = d^{n+1}_{n+1+i}$ for any $i \geq 0$

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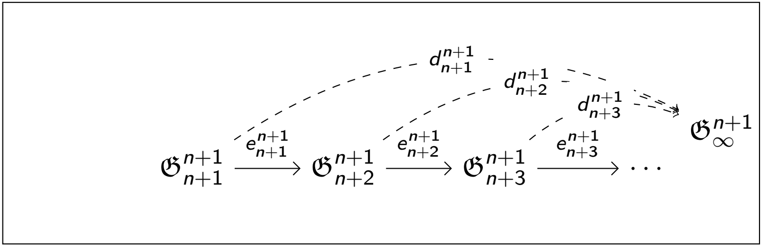

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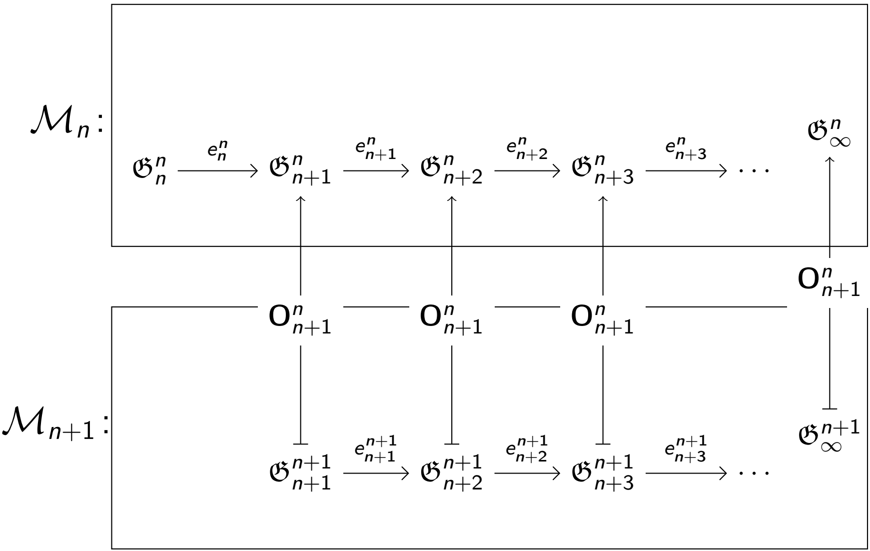

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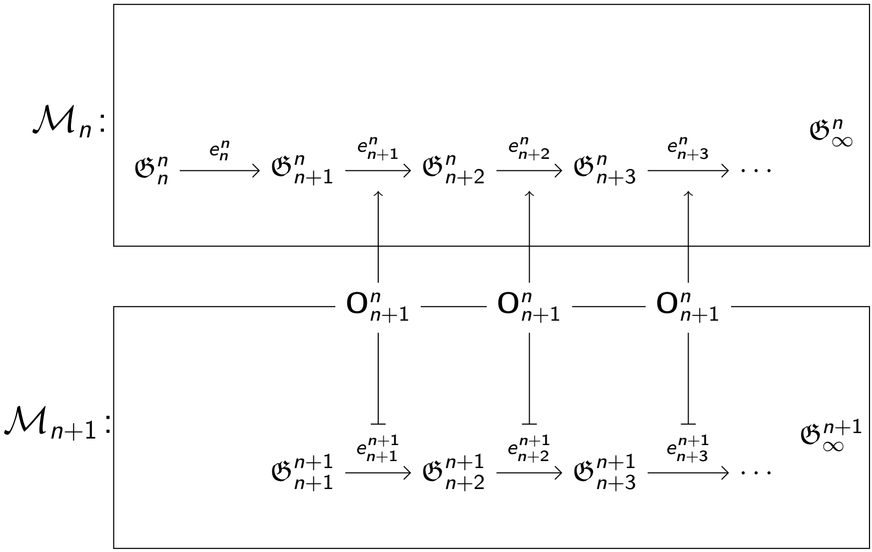

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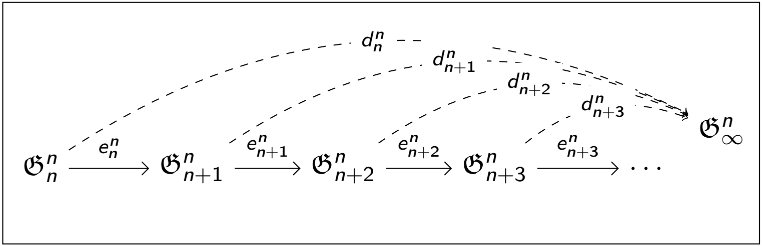
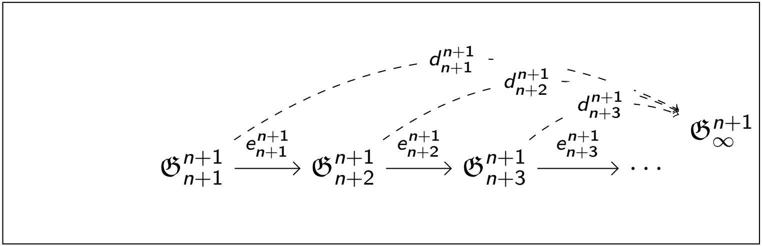
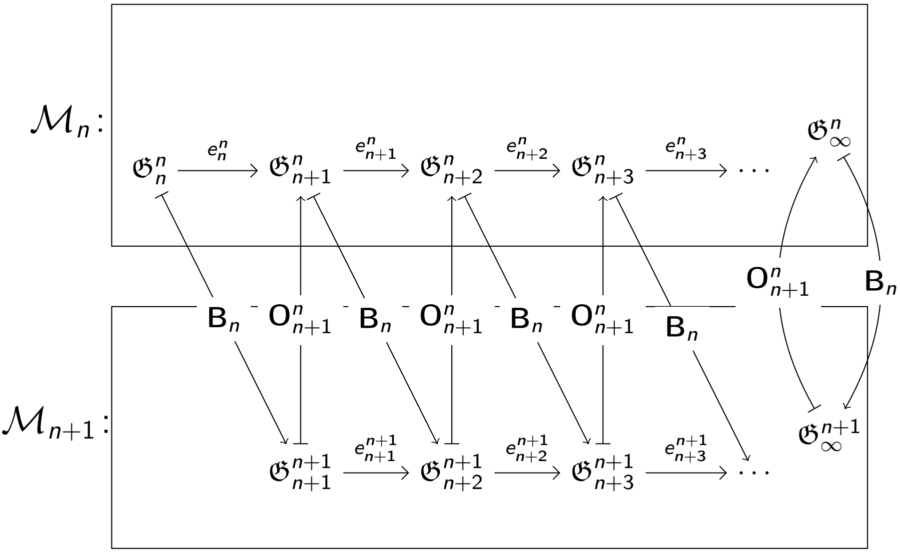


We have $\mathbf{O}_{n+1}^n(d_{n+1+i}^{n+1}) = d_{n+i}^n$ for any $i \geq 0$

$\mathcal{M}_{n+1}$:



Correspondence between Diagrams



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$$\cdots$$

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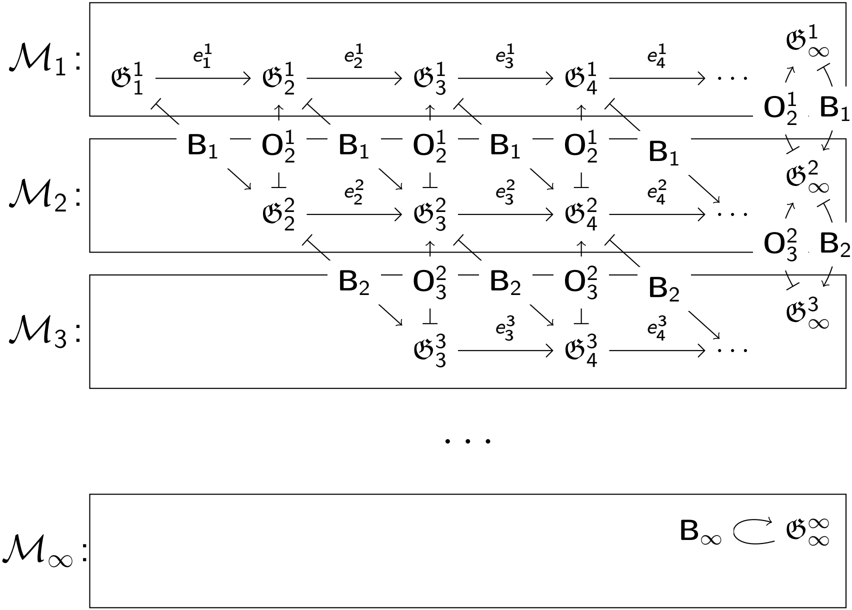

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$\mathcal{L}_n$ is the set of all first order propositions in signature $\mathbf{1}, \mathbf{0}, \wedge, \vee, \sim, [0], \ldots, [n-1]$.

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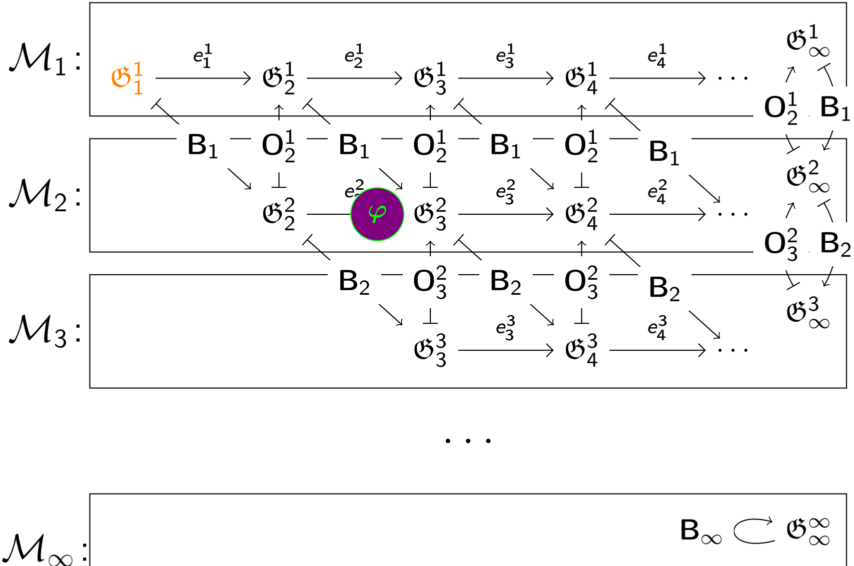
Recall

Theorem (S. Artemov, L. Beklemishev)

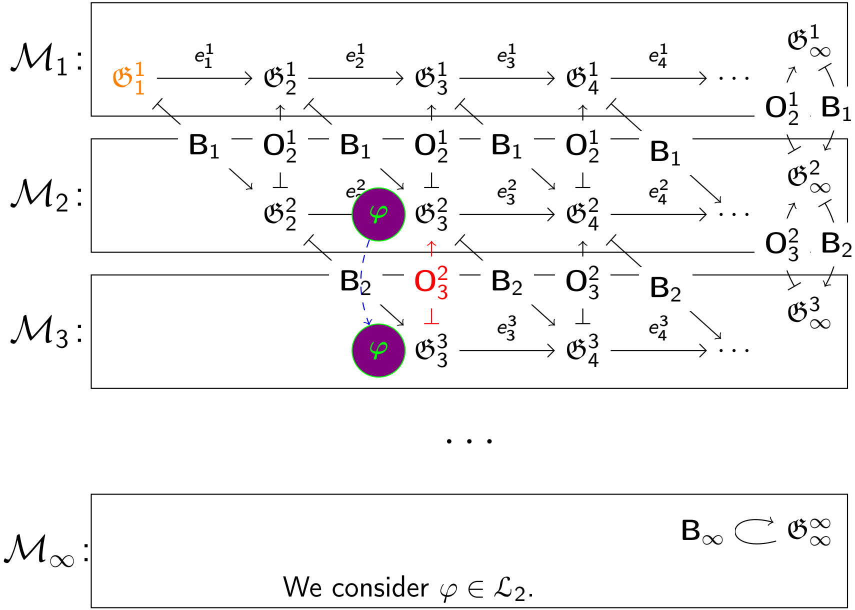
$\mathrm{Th}(\mathfrak{G}_1^1)$ *is decidable.*

Proof of the Theorem



For example we will prove decidability of $\mathrm{Th}(\mathfrak{G}_3^2)$.

Proof of the Theorem



We consider $\varphi \in \mathcal{L}_2$.

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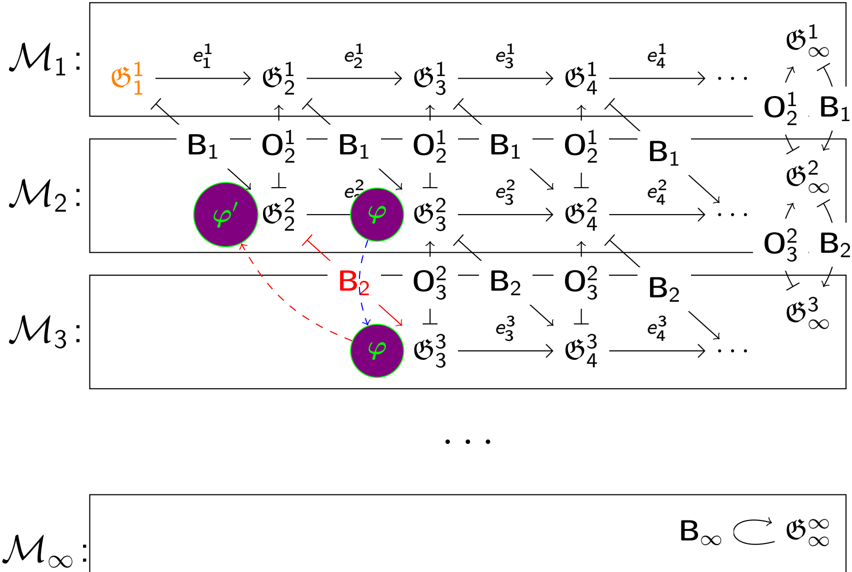

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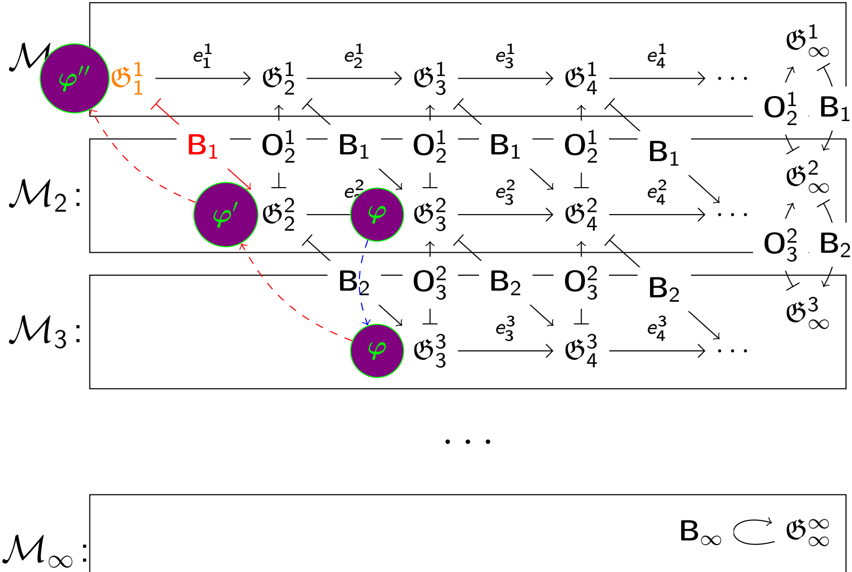

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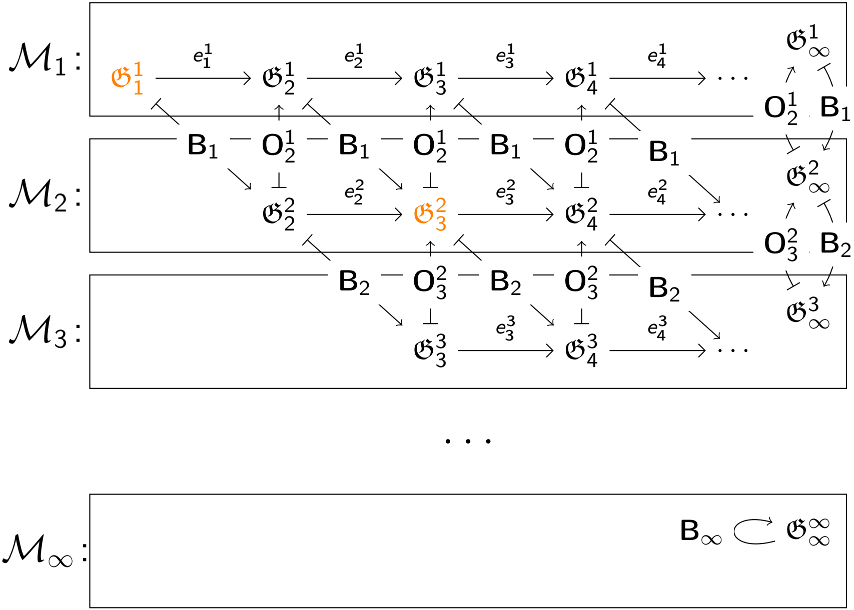

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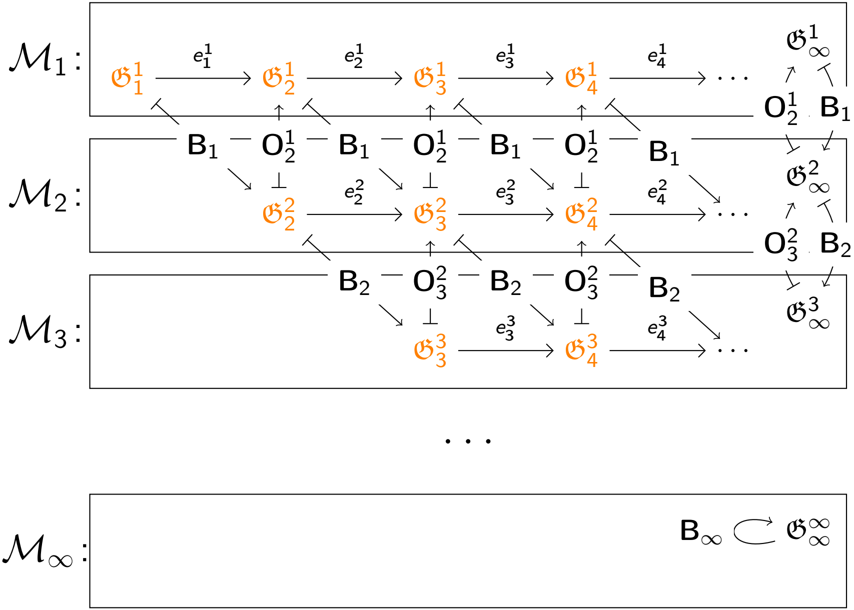

n-Elementary Embeddings

First order formula φ lies in Π_n ($n \geq 0$) if there exists quantifier free ψ such that

$$\varphi \text{ is graphicly equal to } \underbrace{\forall \overline{y}_1 \exists \overline{y}_2 \ldots Q\overline{y}_n}_{n \text{ quantifiers}}(\psi).$$

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Theory $\mathrm{Th}(\mathfrak{G}_\infty^1)$ is decidable.

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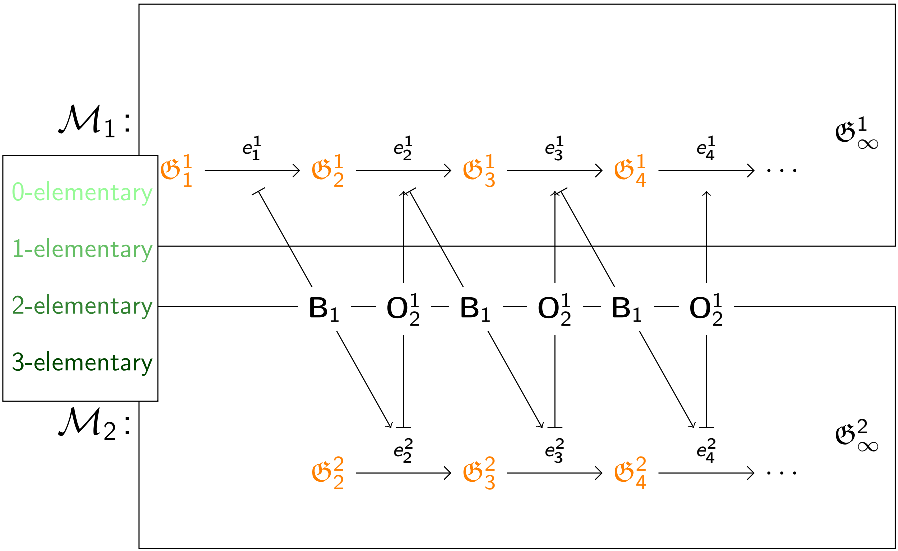

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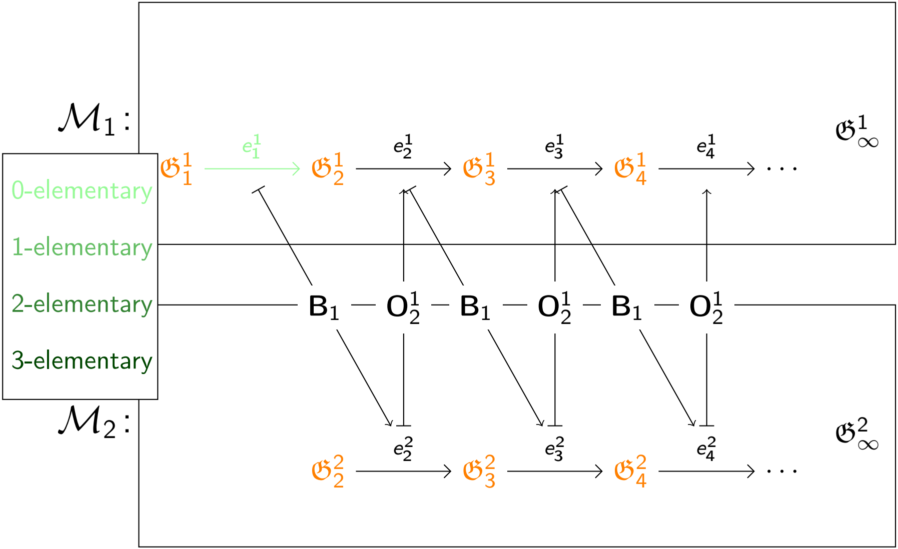

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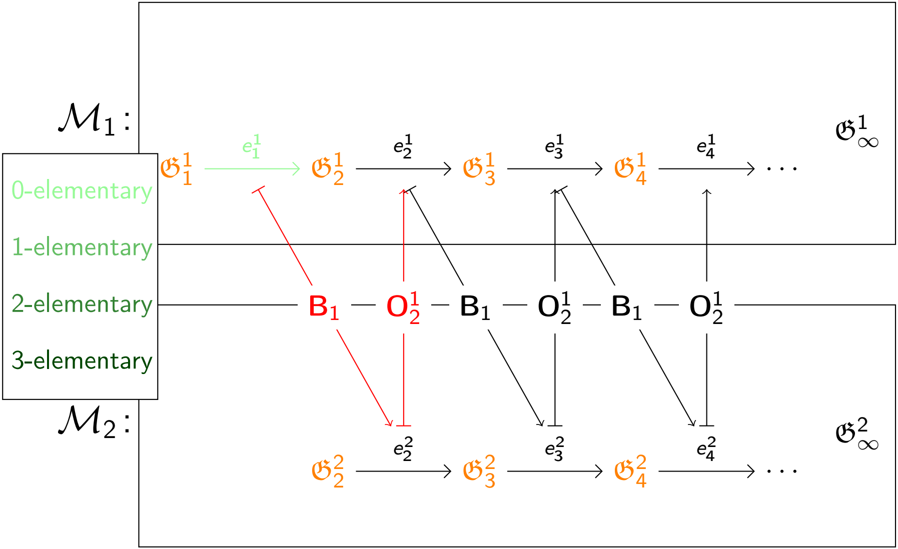

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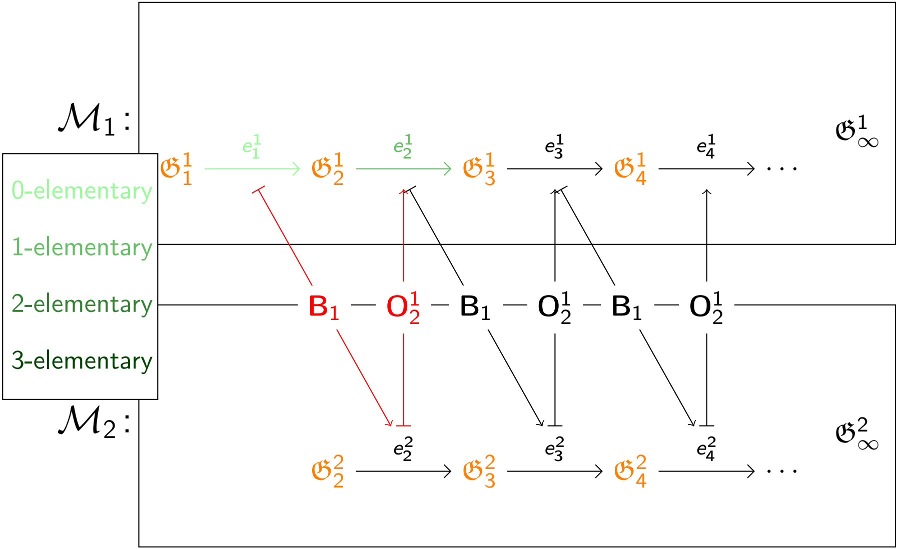

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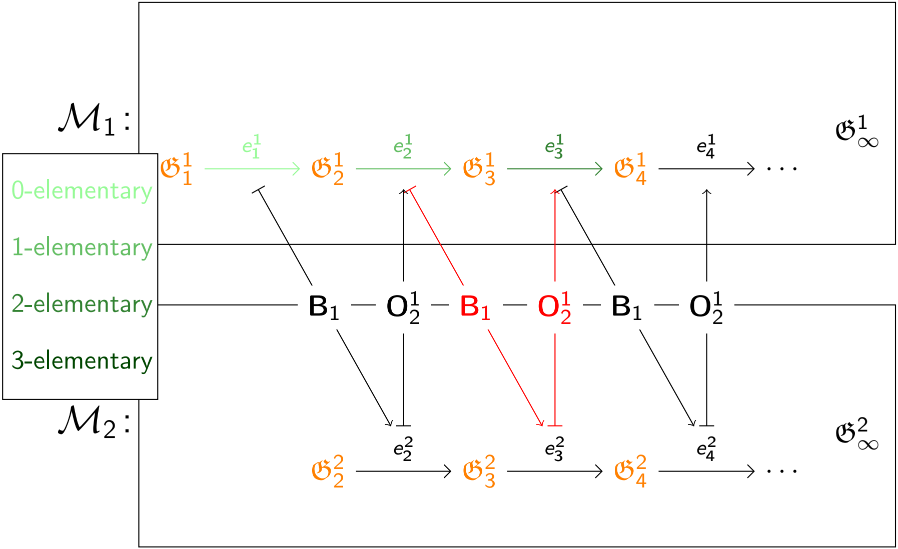

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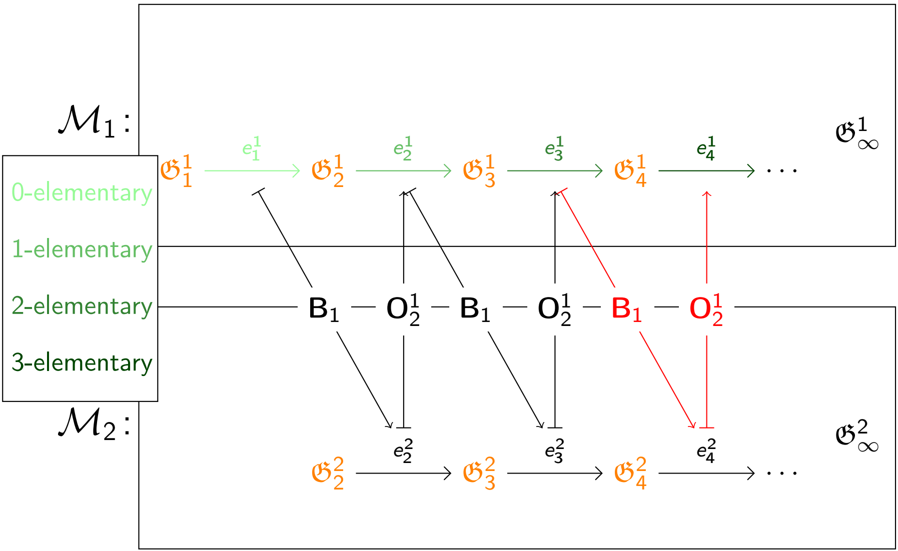

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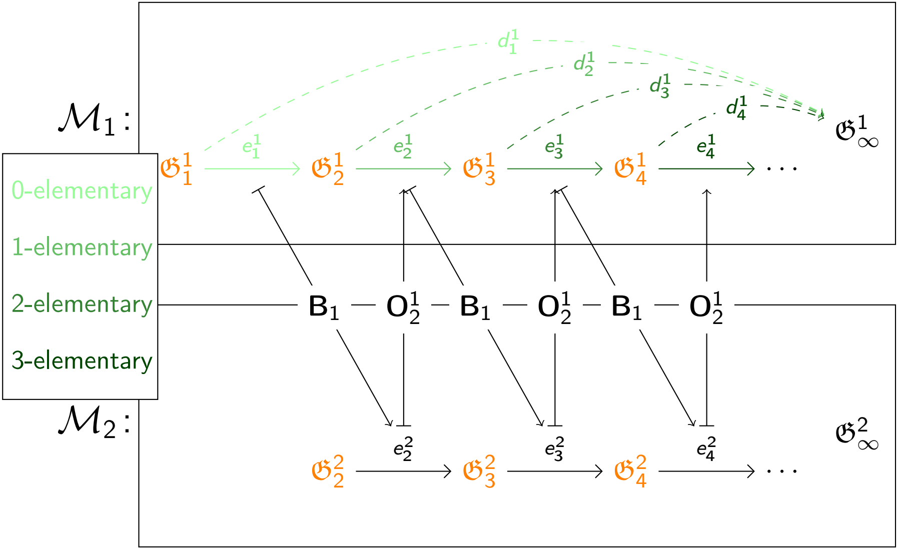

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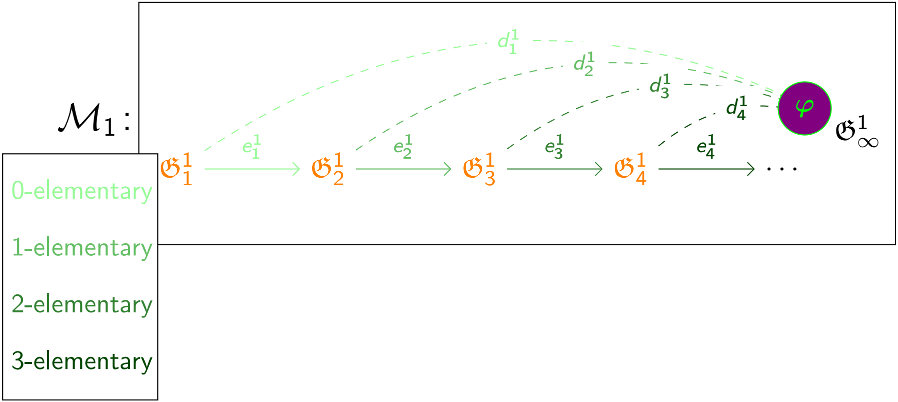

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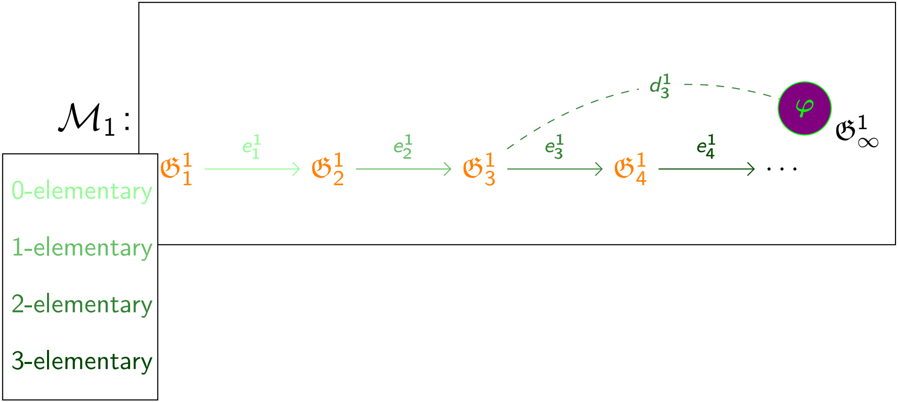

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For example assume that φ has ≤ 2 quantifiers.

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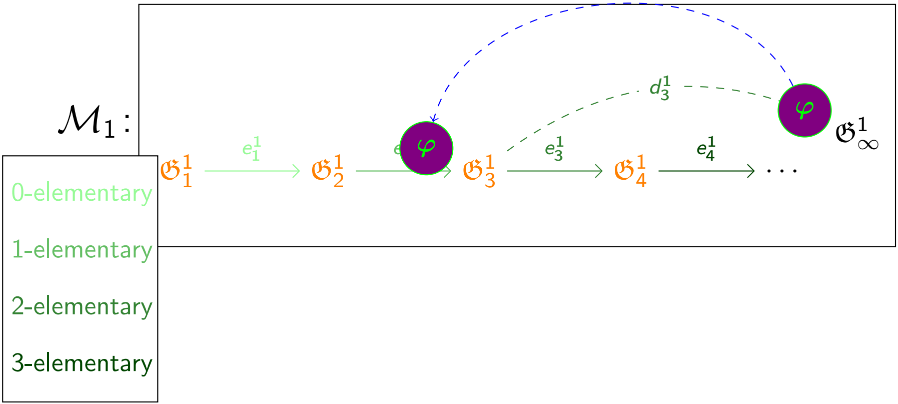

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- 0-elementary
- 1-elementary
- 2-elementary
- 3-elementary

Elementary Theory of $\mathfrak{G}_{\infty}^{\infty}$

Now we will prove our main result

Theorem

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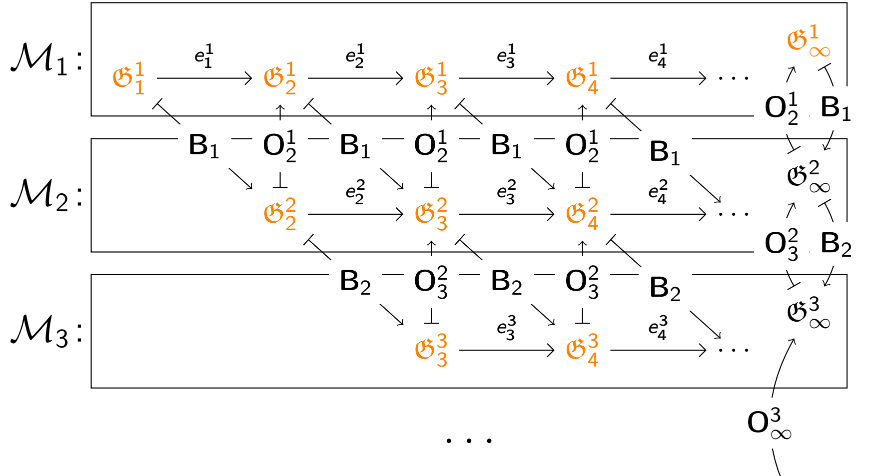

$\mathcal{M}_\infty$: We consider $\varphi \in \mathcal{L}_\infty = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n$. $B_\infty \circlearrowright \mathfrak{G}_\infty^\infty$ φ

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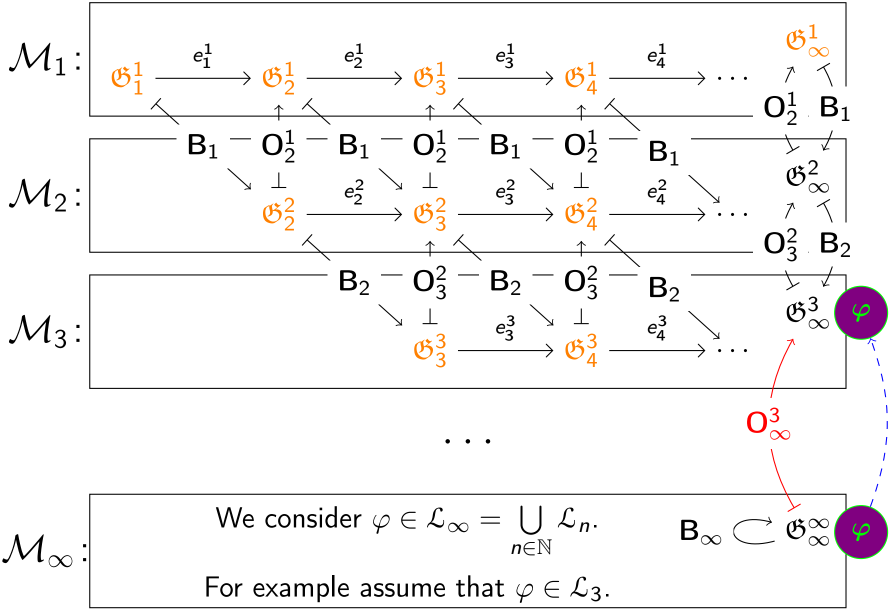

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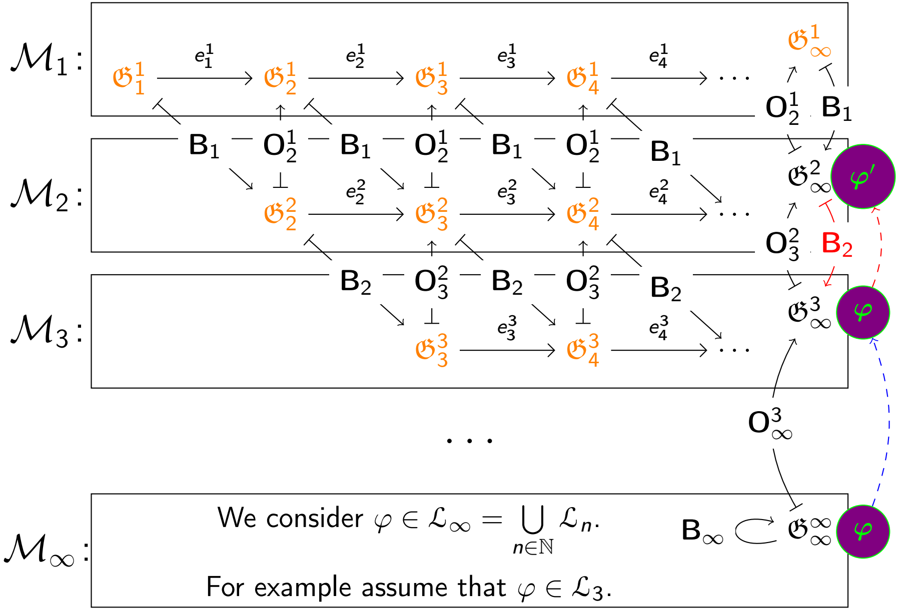

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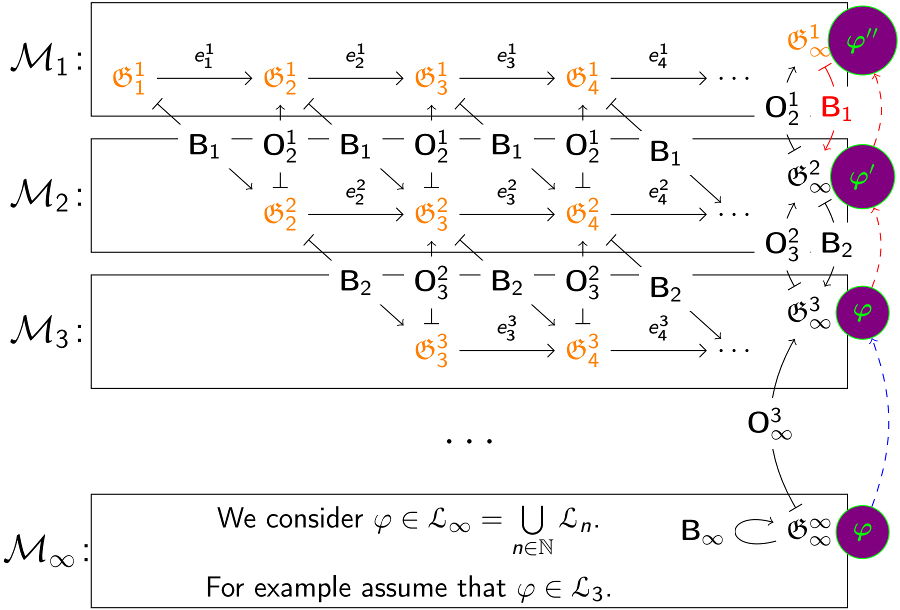

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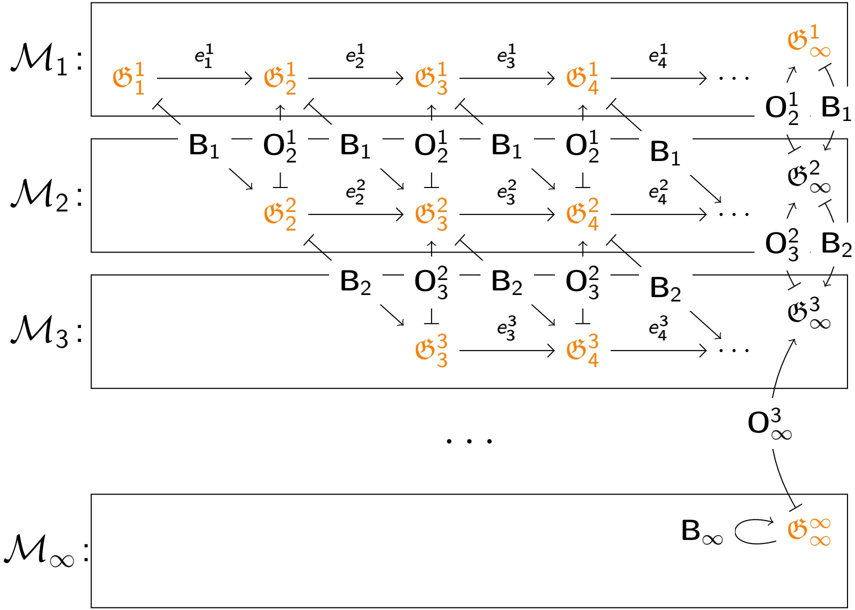

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