Impacts of Reflection Principle

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- higher order number/set theory (highest/lower=2nd/1st)

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We focus on the last aspect (and the 2nd, b/c related).

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I. General Second Order System $\mathbf{BT}^2$

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II. General Results in General $\mathbf{BT}^2$

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VI. Further Task and Conclusion

I. General Second Order System $\mathbf{BT}^2$

Requirement on our Base Theory $\mathbf{BT}^2$

As convention, upper/lower cases for 2nd/1st order.

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3. Finite seq.s for 1st order are coded in a Δ^1_0 way:
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4. There is a class Δ^0_0 of elementary formulae such that
   - Δ^0_0 contains all quantifier-free formulae;
   - pairs $\langle$-, -$\rangle$ are codable in a Δ^0_0 way;
   - Δ^0_0 is closed under $(\exists y)(z = \langle x, y\rangle \wedge \ldots)$ etc.;
   - there is a universal Σ^0_1-formula (Σ^0_n def'd acc.ly).

Instances of $\mathbf{BT}^2$

In second order number/set theory,

- $\mathbf{BT}^2$ is $\mathbf{ACA}_0$ (for number);
  (for set) $\mathbf{NBG}$ (w/o GC) or $\mathbf{NBGW}$ (w/ pred. for GW);
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- pair: $\langle x^{k+1}, y^{k+1}\rangle = \{\langle u^k, v^k\rangle \mid u^k \in x^{k+1} \& v^k \in y^{k+1}\}$.

Formalization of Well-foundedness

Our choice of formalization is:

- $\mathrm{WF}(W) \equiv (\forall X)\mathrm{TI}[\in X](W)$, where
- $\mathrm{TI}[\varphi](W) \equiv (\forall x)((W)_x \subset \{y \mid \varphi(y)\}) \to (\forall x)\varphi(x)$;
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With (suitable form of) choice, $\mathrm{WF}(R)$ is equiv. to

- non-existence of R-decreasing sequence of length ω.

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But this does not allow recursion!

- $\neg\mathrm{WF}(R) \wedge (\forall \vec{X})(\exists H)\mathrm{Hier}[\varphi(\text{-}, \vec{X})](H, R) \to \bot$!

II. General Results in General BT^2

General Results in $\mathbf{BT}^2$

By the same proof as in SONT, we have:

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- $\mathrm{BT}^2 + \Pi^1_{n+1}$-**Red** $\vdash \Pi^1_n$-**TR**; $\mathrm{BT}^2 + \Pi^1_n$-**FP** $\vdash \Pi^1_n$-**TR**;

$\mathit{\Gamma}$-**Red** $\forall x(\varphi(x) \vee \psi(x)) \to \exists X \forall x(\neg\psi(x) \to x \in X \to \varphi(x))$;
$\mathit{\Gamma}$-**TR** $\mathrm{WF}(R) \to (\exists H)\mathrm{Hier}[\varphi](H, R)$;
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Moreover, we can also show

- $\mathrm{BT}^2 + \Pi^0_1$-**LFP** $\vdash \Delta^1_0$-**FP** $\leftrightarrow \Delta^1_0$-**LFP**.

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- $\mathbf{BT}^2 \vdash \Pi^1_1$-**CA** $\leftrightarrow \Delta^1_0$-**LFP**;
- $\mathbf{BT}^2 + \Pi^0_1$-**LFP** $\vdash \Delta^1_0$-**LFP**.

Actually, in all the instances we listed except $\mathbf{ACA}_0$,

- $\mathbf{BT}^2 W \vdash \mathrm{WO}(\varepsilon_W)$, $\mathrm{WO}(\Gamma_W)$ and more;
- $\mathbf{BT}^2 + \Pi^1_1$-**Red** $\vdash \mathrm{Con}_{\mathbf{BT}^2}(\Delta^1_0$-**FP**$)$;
  $\mathbf{BT}^2 + \Delta^1_0$-**FP** $\vdash \mathrm{Con}_{\mathbf{BT}^2}(\Delta^1_0$-**TR**$)$;
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- $\mathbf{BT}^2 \vdash \Pi^0_1$-**LFP**.

III. Our Formulation of Reflection

Additional Assumption for Reflection Principle

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- $(\exists u^k)(x^{k+1} = (y^{k+1})_u) \vee x \in y$ in the higher order $((Qx^{k+1} \epsilon y^{k+1})\varphi(x, y)$ is $(Qu^k)\varphi((y)_u, x)$... etc.).

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For transitive a (i.e., $z \mathbin{\epsilon} y \mathbin{\epsilon} a \to z \mathbin{\epsilon} a$),

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Note: $\varphi^a(X)$ is equiv. to $\varphi^a(X \cap a)$ (if "$X \cap a$" exists).

Reflection Principle

Now, our reflection principle is formulated as:

Δ^1_0**-Ref** $(\forall x)(\exists a)[a$: trans. $\wedge\ x \epsilon a \wedge (\forall z \epsilon a)(\varphi(z) \leftrightarrow \varphi^a(z))$
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Then the last clause implies:

- $(QX)\varphi^a(X)$ is equiv. to $(Qx)\varphi^a(\{u \mid u \,\epsilon\, x\})$; and so
- $(\Pi^1_\infty$-**CA**$)^a$ holds!

Reflection in Instances

In second order number theory,

- even Σ^0_1-reflection does not hold, e.g., $(\exists y)(y > x)$: $(\forall x)(\exists y)(y > x)$ implies $(\forall x < a)(\exists y < a)(y > x)$!

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This is from the following "ontological" difference:

w/o 1st order infinity: SONT (infinity exists only as SO);

with 1st order infinity: SOST, HONT, HOST (it exists as FO).

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In higher order number/set theory,

- if we assume $\mathbf{GW}$, Δ^1_0**-Ref** holds (by Skolemization).

This is from the following “ontological” difference:

w/o 1st order infinity: SONT (infinity exists only as SO);

with 1st order infinity: SOST, HONT, HOST (it exists as FO).

This is why SONT is exceptional among SO systems.

IV. Impacts of Reflection

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For any BT^2 with $\mathrm{BT}^2 \vdash \Delta^1_0$**-Ref**,

- $\mathrm{BT}^2 + \Pi^1_1$**-Red** $\vdash \mathrm{Con}_{\mathbf{BT}^2}(\Delta^1_0$**-FP**$)$;
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which are known to hold for $\mathbf{BT}^2 = \mathbf{ACA}_0$.

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Clearly $\Gamma(F) \subset F$ and $\Gamma^2(F) \subset \Gamma(F)$. By Claim, $F \subset \Gamma(F)$. $\square$

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Ordinal analysis for SONT seems to depend on the specific characters of ω.

Sketch of $\mathrm{BT}^2 + \Pi^0_1$-LFP $\vdash \Delta^1_0$-FP $\leftrightarrow \Delta^1_0$-LFP

Recall the definition of stage comparison $\prec_\Gamma$:

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Proof: The acc. part of a fix.pt. of $(\Gamma')^2$ is $\prec_\Gamma$ (rest.)!

ID_1 and $\widehat{ID}_1$ are equivalent!

For a SO variable X, $\mathbf{BT}^1[X]$ is the $\mathcal{L}^1[X]$-part of $\mathbf{BT}^2$:

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We can define ID_1 and $\widehat{ID}_1$ over $\mathbf{BT}^1$, instead of $\mathbf{PA}$:

- for any $\mathcal{L}^1[X]$-formula $\varphi(z, X^+)$, introduce new F_φ;
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The difference: WF is Π^1_1 (in SONT) or Δ^1_0 (in the others).

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In general, $m+1$-fold AP is stronger than m-fold AP.

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By modifying the famous proof of Δ^1_0-**FP** $\rightarrow$ Δ^1_0-**TR**,

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This shows the exceptional status of ω among infinities (and is among consequences of reflection principle!)

VI. Futher Task and Conclusion

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We can, to some extent, avoid such "controversies" by:

equiconsistency (or proof-theoretic equivalence) b/w the systems with and without these axioms.

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Thank you for your attention!