

Interpolation Properties for Provability Logics GL and GLP

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Craig interpolation

Craig interpolation

If $L \vdash \varphi \rightarrow \psi$, then there is a formula θ containing only common variables of φ and ψ such that $L \vdash \varphi \rightarrow \theta$, $L \vdash \psi \rightarrow \theta$.

Lyndon interpolation

is a strengthening of the Craig one by the additional requirement:
every propositional variable that has a positive (negative) occurrence in θ must also have positive (negative) occurrences both in φ and ψ .

Uniform interpolation

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For any formula φ and any propositional variable p , there is a formula θ such that $PV(\theta) \subset PV(\varphi) \setminus \{p\}$ and for all formulas ψ with $p \notin PV(\psi)$ one has $L \vdash \varphi \rightarrow \psi \iff L \vdash \theta \rightarrow \psi$.

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Theorem (Pitts 1992)

Intuitionistic propositional logic enjoys the uniform interpolation.

Examples

The uniform interpolation holds for:

- ▶ Classical propositional logic
the uniform interpolant for $\varphi(p)$ is $\varphi(\perp) \vee \varphi(\top)$
- ▶ K, T, GL, Grz, S5
- ▶ μ K

Interpolation is not uniform for:

- ▶ K4, S4
- ▶ First order logic
- ▶ J

Gödel-Löb logic

Axioms

- ▶ Boolean tautologies
- ▶ $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- ▶ $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$ (Löb's axiom)

Rules

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi} \qquad \frac{\varphi}{\Box\varphi}$$

Interpolation properties

- ▶ Craig (Smoryński 1978, Boolos 1979)

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Interpolation properties

- ▶ Craig (Smoryński 1978, Boolos 1979)
- ▶ Uniform (Shavrukov 1993)
- ▶ Lyndon

Japaridze polymodal logic GLP

Axioms

- ▶ Boolean tautologies
- ▶ $[n](\varphi \rightarrow \psi) \rightarrow ([n]\varphi \rightarrow [n]\psi)$
- ▶ $[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi$ (Löb's axiom)
- ▶ $\langle m \rangle \varphi \rightarrow [n]\langle m \rangle \varphi$ for $m < n$
- ▶ $[m]\varphi \rightarrow [n]\varphi$ for $m < n$ (monotonicity axiom)

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$$\frac{\varphi, \varphi \rightarrow \psi}{\psi} \quad \frac{\varphi}{[n]\varphi}$$

Interpolation properties

- ▶ Craig (Ignatiev 1993, Beklemishev 2010)

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Interpolation properties

- ▶ Craig (Ignatiev 1993, Beklemishev 2010)
- ▶ **Uniform**

	GL	GLP
Craig	Smoryński 1978 Boolos 1979	Ignatiev 1993 Beklemishev 2010
Lyndon	+	
Uniform	Shavrukov 1993	+

Uniform interpolation for GLP:

proof sketch

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Equivalent formulation

For any formula φ and any propositional variable p , there is a formula θ such that $PV(\theta) \subset PV(\varphi) \setminus \{p\}$ and

- ▶ $L \vdash \varphi \rightarrow \theta$;
- ▶ if $L \vdash \varphi \rightarrow \psi$ and $p \notin PV(\psi)$, then $L \vdash \theta \rightarrow \psi$.

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The formula θ , if it exists in a logic L , will be denoted by $(\exists p \varphi)_L$.

Proof sketch:

- ▶ reduce GLP to a logic M
- ▶ interpret fragments of M with finite number of modalities in the modal μ -calculus
- ▶ show that the uniform interpolation of the modal μ -calculus implies the same property for M (and thus for GLP)

Proof sketch:

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Definition

M is obtained from GLP by excluding the axiom schema:

- ▶ $\langle m \rangle \varphi \rightarrow [n] \langle m \rangle \varphi$ for $m < n$.

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Lemma

$$\text{GLP} \vdash \varphi \iff \text{M} \vdash C(\varphi) \rightarrow \varphi,$$

where

$$C(\varphi) \equiv \Box \bigwedge_{i < s} (\langle m_i \rangle \varphi_i \rightarrow [m_i + 1] \langle m_i \rangle \varphi_i)$$

and $\langle m_i \rangle \varphi_i$ for $i < s$ are all subformulas of φ of the form $\langle k \rangle \psi$, $k < m(\varphi)$.

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where $\langle m_i \rangle \varphi_i$ for $i < s$ are all subformulas of φ of the form $\langle k \rangle \psi$ (we don't require the condition $k < m(\varphi)$).

Then $(\exists p \varphi)_{\text{GLP}}$ can be defined as $(\exists p (\bar{C}(\varphi) \wedge \varphi))_{\text{M}}$.

Fact (interpolation for M)

If $M \vdash \varphi \rightarrow \psi$, then there is a Lyndon interpolant θ such that $m(\theta) \leq \min\{m(\varphi), m(\psi)\}$.

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$(\exists p \varphi)_M$ is simply $(\exists p \varphi)_{M_{m(\varphi)+1}}$.

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- ▶ interpret fragments of M with finite number of modalities in the modal μ -calculus

Introduction to the μ -calculus

The modal μ -calculus extends the language of modal logic with

$$\mu p. \varphi, \quad \nu p. \varphi \quad (\varphi \text{ is positive in } p)$$

Semantics: we interpret a modal formula $\varphi(p)$ over a Kripke model $\mathcal{M} = (W, \vDash)$ as a monotone operator sending

$$S \subset W \quad \text{to} \quad \{x \in W : (\mathcal{M}[p/S], x) \vDash \varphi\}.$$

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$$\mathcal{M}, x \vDash \mu p. \varphi \quad \iff \quad x \in LFP(\varphi)$$

Examples

- ▶ Reachability: $\mu q. (p \vee \diamond q)$
- ▶ Existence of an infinite path: $\nu q. \diamond q$
- ▶ Existence of a path where p appears infinitely often:
 $\nu q. \mu r. ((p \wedge \diamond q) \vee \diamond r)$

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Example

$K4$ and $S4$ fail to have the uniform interpolation but $\mu K4$ and $\mu S4$ enjoy this property.

Interpretations in μ -calculus (D'Agostino)

Programs

Let φ represents any closed μ -formula, then

$$\alpha = k \mid \varphi? \mid \alpha \cdot \alpha \mid \alpha \cup \alpha \mid \alpha^*$$

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Semantics

Given a Kripke frame $(W; \{R_k\})$, and a program α , then R_α is a binary relation defined as follows:

- ▶ R_k is already defined
- ▶ $R_{\varphi?} = \{(w, w) \in W^2 : w \models \varphi\}$
- ▶ $R_{\alpha\beta} = R_\alpha \circ R_\beta$
- ▶ $R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$
- ▶ $R_{\beta^*} = (R_\beta)^*$.

Now we define an abbreviation $[\alpha]\varphi$:

- ▶ $[k]\psi$ is already defined
- ▶ $[\varphi?]\psi \equiv \varphi \rightarrow \psi$
- ▶ $[\alpha\beta]\psi \equiv [\alpha][\beta]\psi$
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Fact

For any Kripke model \mathcal{M} we have

$$\mathcal{M}, x \vDash [\alpha]\varphi \iff \forall y (xR_\alpha y \Rightarrow \mathcal{M}, y \vDash \varphi).$$

Interpretation

Let $\pi : \{\text{modalities}\} \longrightarrow \{\text{programs}\}$.

For a μ -formula φ , we replace all occurrences of $[k]$ by $[\pi(k)]$ and denote the result by $\pi(\varphi)$.

For a Kripke frame $\mathcal{W} = (W; \{R_k\})$, define $\mathcal{W}_\pi = (W; \{R_{\pi(k)}\})$.

Analogously for Kripke models.

Fact

For any Kripke model \mathcal{M} we have

$$\mathcal{M}_\pi, x \vDash \varphi \iff \mathcal{M}, x \vDash \pi(\varphi).$$

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- ▶ $\mu\text{S4} \vdash \varphi \leftrightarrow \pi(\varphi)$;
- ▶ $\mu\text{S4} \vdash \varphi \Leftrightarrow \mu\text{K} \vdash \pi(\varphi)$.

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Now $(\exists p \varphi)_{\mu\text{S4}}$ can be defined as $(\exists p \pi(\varphi))_{\mu\text{K}}$

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Thus GL has the uniform interpolation

M_n with n modalities $\{0, 1, \dots, n - 1\}$

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Let $\pi(k) = (\mu p. [\bar{0}]p)?(\bar{k})^+$,

where $\bar{k} = k \cup (k+1) \cup \dots \cup (n-1)$.

M_n with n modalities $\{0, 1, \dots, n-1\}$

Let $\pi(k) = (\mu p. [\bar{0}]p)?(\bar{k})^+$,

where $\bar{k} = k \cup (k+1) \cup \dots \cup (n-1)$.

Lemma

We have

- ▶ \mathcal{W}_π is a M_n -frame for any \mathcal{W} ;
- ▶ \mathcal{W} is a M_n -frame $\Rightarrow \mathcal{W}_\pi = \mathcal{W}$.

Lemma

We have

- ▶ $M_n \vdash \varphi \leftrightarrow \pi(\varphi)$;
- ▶ $M_n \vdash \varphi \Leftrightarrow \mu K_n \vdash \pi(\varphi)$.

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Fact (μ -elimination)

$M_n = \mu M_n$; i.e., every μ -formula φ is equivalent in μM to a μ -free formula ψ such that $PV(\psi) \subset PV(\varphi)$ and $m(\psi) \leq m(\varphi)$.

Lemma

For any n the system M_n enjoys uniform interpolation.

Proof

Let us check that $(\exists p \varphi)_{M_n}$ can be defined as $(\exists p \pi(\varphi))_{\mu K_n}$. Here π is the constructed interpretation of M_n in μK_n .

$$\mu K_n \vdash \pi(\varphi) \rightarrow (\exists p \pi(\varphi))_{\mu K_n}, \quad M_n \vdash \varphi \leftrightarrow \pi(\varphi), \quad \mu K_n \subset \mu M_n = M_n.$$

We get

$$M_n \vdash \varphi \rightarrow (\exists p \pi(\varphi))_{\mu K_n}.$$

If $M_n \vdash \varphi \rightarrow \psi$ and $p \notin PV(\psi)$, then

$$\mu K_n \vdash \pi(\varphi) \rightarrow \pi(\psi), \quad \mu K_n \vdash (\exists p \pi(\varphi))_{\mu K_n} \rightarrow \pi(\psi).$$

Therefore,

$$M_n \vdash (\exists p \pi(\varphi))_{\mu K_n} \rightarrow \psi.$$

q.e.d.

If $M_n \vdash \varphi \rightarrow \psi$ and $p \notin PV(\psi)$, then

$$\mu K_n \vdash \pi(\varphi) \rightarrow \pi(\psi), \quad \mu K_n \vdash (\exists p \pi(\varphi))_{\mu K_n} \rightarrow \pi(\psi).$$

Therefore,

$$M_n \vdash (\exists p \pi(\varphi))_{\mu K_n} \rightarrow \psi.$$

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Theorem

GLP enjoys uniform interpolation.