PSPACE complexity of GLP and some other modal logics

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Japaridze's Polymodal Logic and stratified frames

Theorem (Beklemishev, 2007)

There exists a polynomial-time translation f such that for any formula φ we have

$$\mathrm{GLP} \vdash \varphi \Leftrightarrow f(\varphi) \textit{ is valid in the class of stratified frames.}$$

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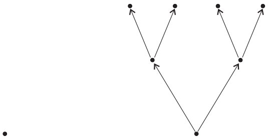

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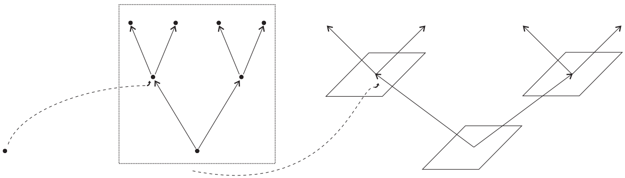

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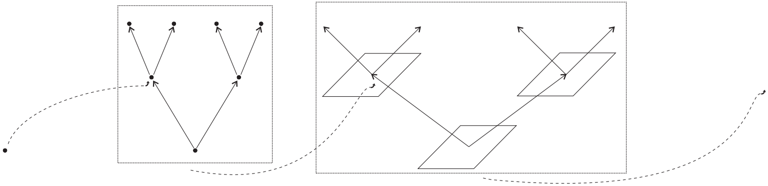

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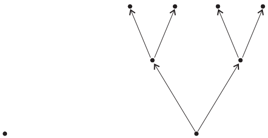

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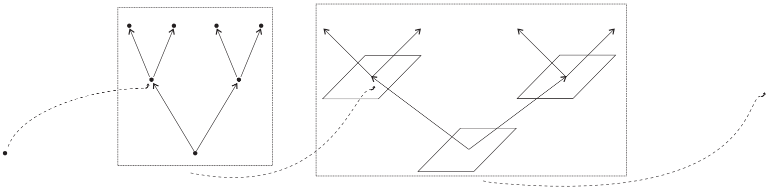

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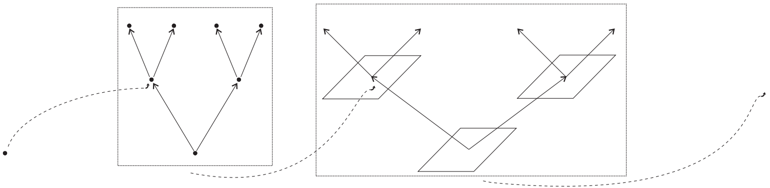

Main idea: if a class of transitive frames is "simple" (e.g. its validity problem is in PSPACE), then the class of ordered sums of these frames over finite partial orders is also "simple".

1. Unimodal case
    - Ordered sums of transitive frames
    - Truth-preserving transformations for ordered sums of frames
    - PSPACE-decidability of ordered sums of frames
2. Polymodal case
    - PSPACE-decidability of GLP

Kripke semantics for propositional modal logics: the unimodal case

Propositional modal formulas are constructed from the countable set of propositional variables $PV = \{p_0, p_1, \dots\}$ using the classical connectives and the unary connectives $\Diamond$ and $\Box$.

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A Kripke frame $\mathsf{F} = (W, R)$ *is a nonempty set* W *with a binary relation* $R \subseteq W \times W$.

A Kripke model $\mathsf{M} = (\mathsf{F}, \theta)$ *is a frame with* a valuation $\theta : PV \to 2^W$.

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φ is true at a point w in M, $\mathsf{M}, w \models \varphi$:

$$
\begin{array}{lcl}
\mathsf{M}, w \models p & \Leftrightarrow & w \in \theta(p); \\
\mathsf{M}, w \models \varphi \wedge \psi & \Leftrightarrow & \mathsf{M}, w \models \varphi \text{ and } \mathsf{M}, w \models \psi; \\
\mathsf{M}, w \models \varphi \vee \psi & \Leftrightarrow & \mathsf{M}, w \models \varphi \text{ or } \mathsf{M}, w \models \psi; \\
\mathsf{M}, w \models \varphi \to \psi & \Leftrightarrow & \mathsf{M}, w \not\models \varphi \text{ or } \mathsf{M}, w \models \psi; \\
\mathsf{M}, w \models \Diamond\varphi & \Leftrightarrow & \exists v(wRv \ \& \ \mathsf{M}, v \models \varphi); \\
\mathsf{M}, w \models \Box\varphi & \Leftrightarrow & \forall v(wRv \Rightarrow \mathsf{M}, v \models \varphi).
\end{array}
$$

φ *is valid in a frame* F (notation: $\mathsf{F} \models \varphi$) iff φ is true at any point in any model based of F.
φ *is valid in a class of frames* $\mathcal{F}$ (notation: $\mathcal{F} \models \varphi$) iff φ is valid in all $\mathsf{F} \in \mathcal{F}$.

Examples.

$(W, R) \models \Diamond\Diamond p \to \Diamond p$ iff R is transitive

$(W, R) \models p \to \Diamond p$ iff R is reflexive

$AxGL = \Box(\Box p \to p) \to \Box p$

$(W, R) \models AxGL$ iff (W, R) is a strict partial order without infinite ascending chains.

A set of modal formulas L is called *a normal logic*, if

- L contains all boolean tautologies
- L contains the formulas $\Box(p \to q) \to (\Box p \to \Box q)$ and $\Diamond p \leftrightarrow \neg\Box\neg p$
- L is closed under Modus Ponens, substitutions, and *Generalization* rule:

$$\text{if } \varphi \in \mathrm{L}, \text{ then } \Box\varphi \in \mathrm{L}$$

For a class of frames $\mathcal{F}$, the set of all valid in $\mathcal{F}$ formulas is called the *logic of* $\mathcal{F}$.
For any class of frames $\mathcal{F}$, its logic is a normal logic; the converse is false.

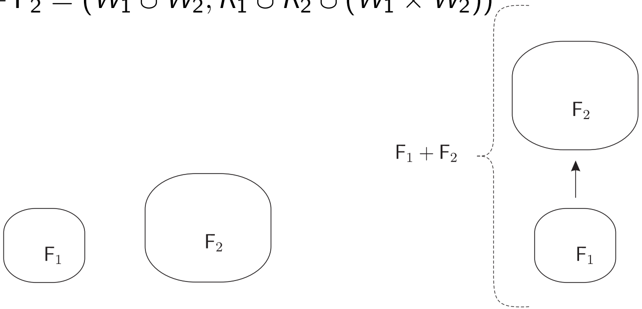
K denotes the least normal modal logic.
For a formula φ and a logic L, $L+\varphi$ is the least logic containing φ and L.

$\mathrm{K4} = K + \Diamond\Diamond p \to \Diamond p$
$\mathrm{S4} = \mathrm{K4} + p \to \Diamond p$
$\mathrm{GL} = \mathrm{K} + \Box(\Box p \to p) \to \Box p$

K is the logic of the class of all (finite) frames.
$\mathrm{K4}$ is the logic of the class of all (finite) transitive frames.
$\mathrm{S4}$ is the logic of the class of all (finite) transitive reflexive frames.
GL is the logic of the class of all finite strict partial orders.

Ordered sums of frames

For $F_1 = (W_1, R_1)$, $F_2 = (W_2, R_2)$, $W_1 \cap W_2 = \varnothing$, put
$F_1 + F_2 = (W_1 \cup W_2, R_1 \cup R_2 \cup (W_1 \times W_2))$



Ordered sums of frames

$\mathsf{I} = (W, \leq)$ is a finite partial order, $W = \{w_1, \dots, w_m\}$
$\mathsf{F}_1 = (V_1, S_1), \dots, \mathsf{F}_m = (V_m, S_m)$ are transitive frames (*bricks*)

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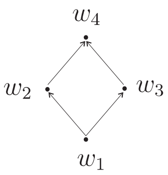

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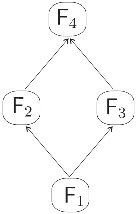

$\mathsf{I}[(\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)]$

$\mathsf{I}[(\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)] = (\overline{W}, \overline{R})$:

$\overline{W} = (\{w_1\} \times V_1) \cup \cdots \cup (\{w_m\} \times V_m)$

$(w_i, v')\overline{R}(w_j, v'') \Leftrightarrow (i \neq j \ \& \ w_i \leq w_j)$ or $(i = j \ \& \ v' S_i v'')$

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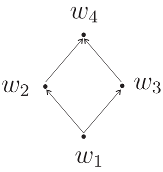

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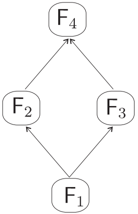


$\mathsf{I}[(\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)]$

For a class $\mathcal{F}$ of frames,

$\sum_{\mathsf{I}} \mathcal{F} = \{\mathsf{I}[(\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)] \mid \mathsf{F}_1, \ldots, \mathsf{F}_m \in \mathcal{F}\}$

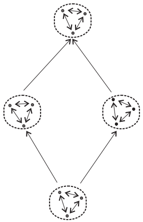
For a class $\mathcal{I}$ of finite partial orders,

$\sum_{\mathcal{I}} \mathcal{F} = \bigcup \{\sum_{\mathsf{I}} \mathcal{F} \mid \mathsf{I} \in \mathcal{I}\}$

Example: transitive frames

Finite clusters: for $n \geq 1$,
put $\mathsf{C}_n = (W_n, W_n \times W_n)$, where $W_n = \{1, \dots, n\}$;
$\mathsf{C}_0 := (\{0\}, \varnothing)$ (*degenerate cluster*).

Every finite transitive frame
is isomorphic to an ordered sum of finite clusters:



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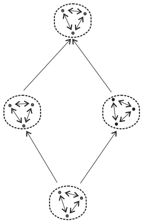

Let $\mathcal{PO}$ denote the class of all finite partial orders.

Then (up to isomorphisms):

$\sum_{\mathcal{PO}} \{\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \ldots\}$ is the class of all finite transitive frames,

$\sum_{\mathcal{PO}} \{\mathsf{C}_1, \mathsf{C}_2, \ldots\}$ is the class of all finite transitive reflexive frames (preorders).

φ *is satisfiable in a frame* F iff $\neg\varphi$ is not valid in F.
φ *is satisfiable in a class of frames* $\mathcal{F}$ iff φ is satisfiable in some F $\in \mathcal{F}$.
For a class of frames $\mathcal{F}$, $\mathcal{F}$-`Sat` denotes the satisfiability problem for $\mathcal{F}$.

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$\mathcal{PO}$ denotes the class of all finite partial orders

Theorem

Let $\mathcal{F}$ be a non-empty cone-closed class of transitive frames.
If $\mathcal{F}$-$\mathcal{S}at$ is in PSPACE, then $\left(\sum_{\mathcal{PO}} \mathcal{F}\right)$-$\mathcal{S}at$ is PSPACE-complete.

Examples

Well-known facts:
the logics $\mathrm{K4}, \mathrm{S4}$, Gödel-Löb logic GL, and Grzegorczyk logic $\textsc{Grz}$ are PSPACE-complete [Ladner, Spaan, ...].

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Another proofs of these facts:
The above logics are PSPACE-complete, since

$$\mathrm{K4} = \mathrm{L}\left(\sum_{\mathcal{PO}}\{\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2 \ldots\}\right),$$

$$\mathrm{S4} = \mathrm{L}\left(\sum_{\mathcal{PO}}\{\mathsf{C}_1, \mathsf{C}_2 \ldots\}\right),$$

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where $\mathsf{C}_0, \mathsf{C}_1, \ldots$ are finite clusters.

$$\mathrm{GL} = \mathrm{L}\left(\sum_{\mathcal{PO}}\{C_0\}\right),$$

$$\mathrm{G{\small RZ}} = \mathrm{L}\left(\sum_{\mathcal{PO}}\{C_1\}\right).$$

Truth-preserving transformations for ordered sums

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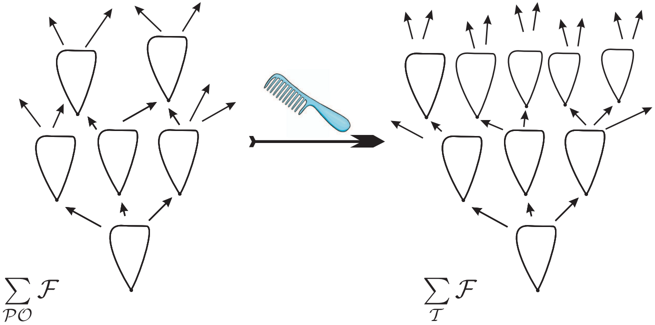
$\mathcal{T}$ denotes the class of all finite transitive trees.

Lemma (via *unravelling*)

For a class $\mathcal{F}$ of transitive frames,

φ *is* $\sum_{\mathcal{PO}} \mathcal{F}$*-satisfiable* $\Longrightarrow$ φ *is* $\sum_{\mathcal{T}} \mathcal{F}$*-satisfiable*



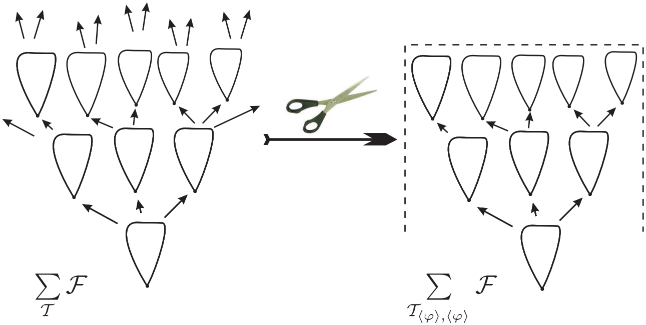
Truth-preserving transformations for ordered sums

$\mathcal{T}_{h,b}$ denotes the class of transitive trees with the height not greater then h and the branching not greater then b.

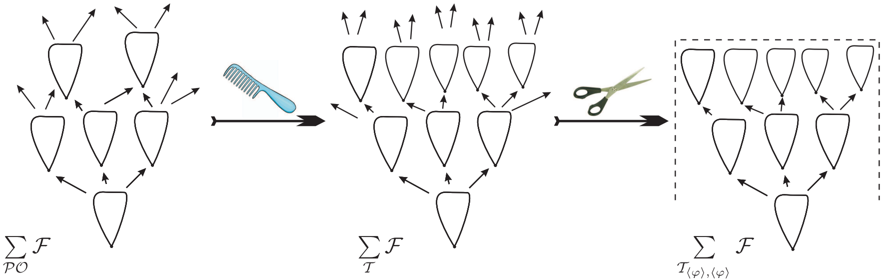
$\langle\varphi\rangle$ denotes the length of φ.

Lemma (via *selective filtration*)

$$\varphi \textit{ is } \sum_{\mathcal{T}} \mathcal{F}\textit{-satisfiable} \quad \Longrightarrow \quad \varphi \textit{ is } \sum_{\mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}} \mathcal{F}\textit{-satisfiable}$$



Good looking frames:



$\mathcal{PO}$ is the class of all finite partial orders

$\mathcal{T}$ is the class of all finite transitive trees

$\mathcal{T}_{h,b} = \{\mathsf{T} \in \mathcal{T} \mid \mathit{height}(\mathsf{T}) \leq h,\ \mathit{branching}(\mathsf{T}) \leq b\}$

$\langle\varphi\rangle$ is the length of φ;

Theorem

For a class $\mathcal{F}$ of transitive frames,

$$\mathrm{L}\left(\sum_{\mathcal{PO}} \mathcal{F}\right) = \mathrm{L}\left(\sum_{\mathcal{T}} \mathcal{F}\right);$$

moreover, for any formula φ,

$$\varphi \textit{ is } \sum_{\mathcal{PO}} \mathcal{F}\textit{-satisfiable} \iff \varphi \textit{ is } \sum_{\mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}} \mathcal{F}\textit{-satisfiable.}$$

Conditional satisfiability

M is a Kripke model

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\begin{array}{lcl}
M, w \nvDash \bot; & & \\
M, w \vDash p & \Leftrightarrow & w \in \theta(p); \\
M, w \vDash \varphi \to \psi & \Leftrightarrow & M, w \nvDash \varphi \text{ or } M, w \vDash \psi; \\
M, w \vDash \Diamond\varphi & \Leftrightarrow & \exists v (wRv \ \& \ M, v \vDash \varphi).
\end{array}
$$

"φ is true at w in M".

Conditional satisfiability

M is a Kripke model, Ψ is a set of formulas

$$
\begin{array}{lcl}
\mathsf{M}|\Psi, w \not\models \bot & & \\
\mathsf{M}|\Psi, w \models p & \Leftrightarrow & w \in \theta(p); \\
\mathsf{M}|\Psi, w \models \varphi \to \psi & \Leftrightarrow & \mathsf{M}|\Psi, w \not\models \varphi \text{ or } \mathsf{M}|\Psi, w \models \psi \\
\mathsf{M}|\Psi, w \models \Diamond\varphi & \Leftrightarrow & \exists v (wRv \;\&\; \mathsf{M}|\Psi, v \models \varphi) \\
 & & \text{ or } \varphi \in \Psi \text{ or } \Diamond\varphi \in \Psi
\end{array}
$$

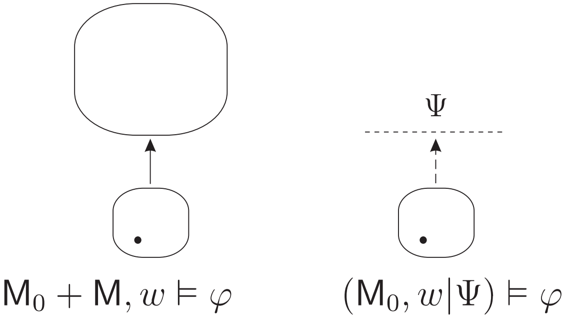
"φ *is true at* w *in* M *under the condition* Ψ ".

Lemma

Consider transitive models M_0, M, *their ordered sum* $\mathsf{M}_0 + \mathsf{M}$, *and a formula* φ. *Put*

$$\Psi = \{\psi \in Sub(\varphi) \mid \mathsf{M}, v \vDash \psi \textit{ for some } v\}.$$

Then for any $w \in \mathsf{M}_0$, $\mathsf{M}_0 + \mathsf{M}, w \vDash \varphi \Leftrightarrow \mathsf{M}_0|\Psi, w \vDash \varphi$.



Lemma

Let $\mathcal{F}, \mathcal{G}$ be cone-closed classes of transitive rooted frames.
If $\mathcal{F}$-$\mathcal{S}at$, $\mathcal{G}$-$\mathcal{S}at$ are in PSPACE, then $(\mathcal{F} + \mathcal{G})$-$\mathcal{S}at$ is in PSPACE.

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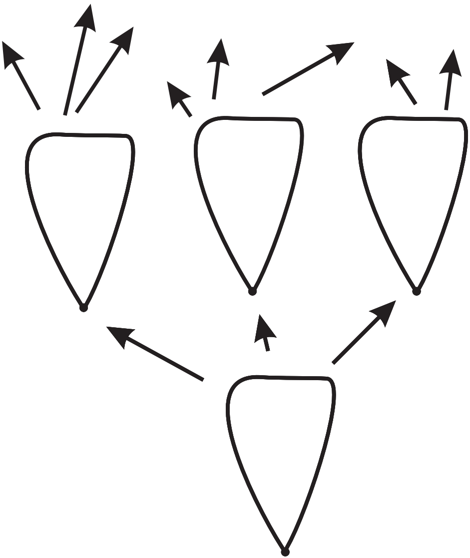
Note that if $\mathsf{G} \in \sum_{\mathcal{T}_{h+1,b}} \mathcal{F}$ for some $h, b \geq 1$, then G is either isomorphic to a frame $\mathsf{F} \in \mathcal{F}$ or isomorphic to a frame $\mathsf{F} + (\mathsf{G}_1 \sqcup \cdots \sqcup \mathsf{G}_{b'})$, where $1 \leq b' \leq b$, $\mathsf{F} \in \mathcal{F}$, $\mathsf{G}_1, \ldots \mathsf{G}_{b'} \in \sum_{\mathcal{T}_{h,b}} \mathcal{F}$.

Satisfiability on tree-like frames



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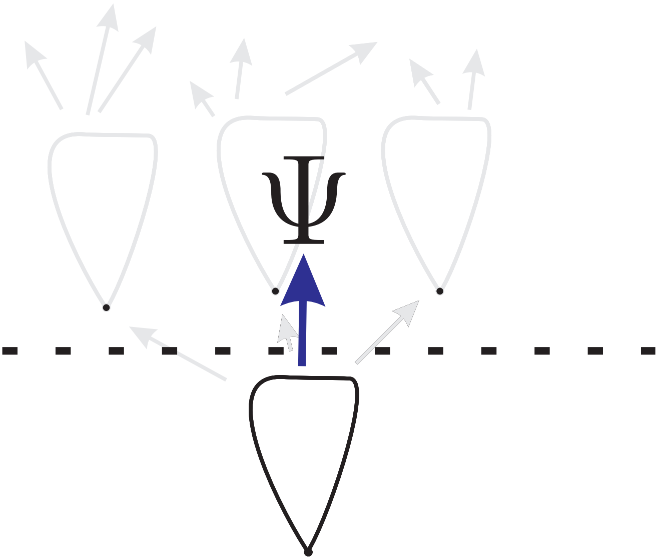

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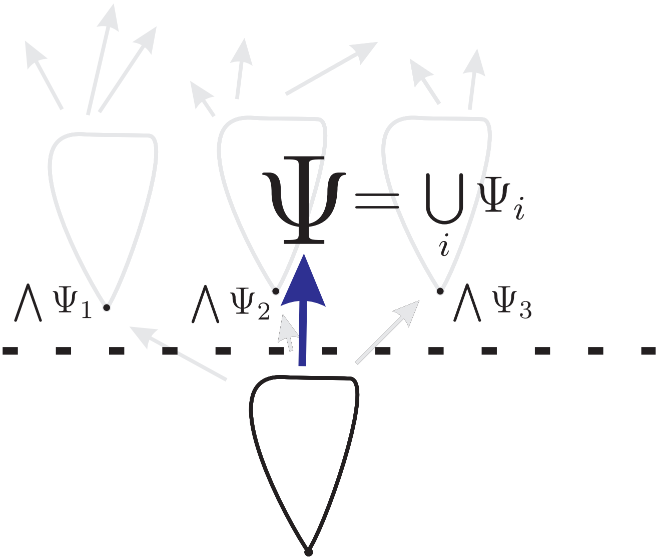

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That is

$$\mathrm{S4.2} = \mathrm{L}(\sum_{\mathcal{PO}} \mathcal{G} + \mathcal{G}),$$

where $\mathcal{G} = \{\mathsf{C}_1, \mathsf{C}_2, \dots\}$.

Example: logic LM

$\mathrm{LM} = \mathrm{K4} + \Diamond\top + \Diamond p_1 \wedge \Diamond p_2 \to \Diamond(\Diamond p_1 \wedge \Diamond p_2)$
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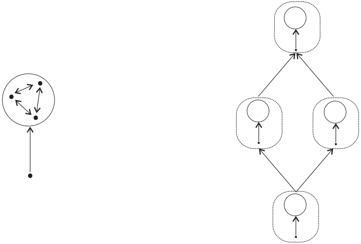

LM is complete w.r.t. the class of its finite frames [Shehtman, Sh., 2002], thus $\mathrm{LM} = \mathrm{L}(\sum_{\mathcal{PO}}((\{\mathsf{C}_0\} + \mathcal{G}) \cup \mathcal{G}))$, where $\mathcal{G} = \{\mathsf{C}_1, \mathsf{C}_2, \ldots\}$.

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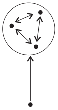

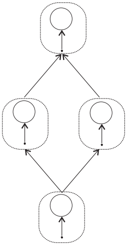

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$\{\mathsf{C}_0\}$-$\mathtt{Sat}$, $\mathcal{G}$-$\mathtt{Sat}$ are in PSPACE (more precisely, in NP), so
$(\{\mathsf{C}_0\} + \mathcal{G})$-$\mathtt{Sat}$ is in PSPACE, so $((\{\mathsf{C}_0\} + \mathcal{G}) \cup \mathcal{G})$-$\mathtt{Sat}$ is in PSPACE.
Finally, $\sum_{\mathcal{PO}}((\{\mathsf{C}_0\} + \mathcal{G}) \cup \mathcal{G})$-$\mathtt{Sat}$ is in PSPACE.

Kripke semantics for propositional modal logics: the polymodal case

Propositional modal formulas are constructed from the countable set of propositional variables $PV = \{p_0, p_1, \dots\}$ using the classical connectives and the unary connectives $\Diamond_i$ and $\Box_i$, $i \in I$.

A Kripke frame $\mathsf{F} = (W, \{R_i\}_{i \in I})$ *is a nonempty set* W *with binary relations* $R_i \subseteq W \times W$*.*

A Kripke model $\mathsf{M} = (\mathsf{F}, \theta)$ *is a frame with* a valuation $\theta : PV \to 2^W$*.*

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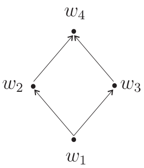
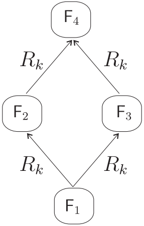
φ is true at a point w in M, $\mathsf{M}, w \vDash \varphi$:

$$
\begin{array}{lcl}
\mathsf{M}, w \vDash p & \Leftrightarrow & w \in \theta(p); \\
\mathsf{M}, w \vDash \varphi \wedge \psi & \Leftrightarrow & \mathsf{M}, w \vDash \varphi \text{ and } \mathsf{M}, w \vDash \psi; \\
\mathsf{M}, w \vDash \varphi \vee \psi & \Leftrightarrow & \mathsf{M}, w \vDash \varphi \text{ or } \mathsf{M}, w \vDash \psi; \\
\mathsf{M}, w \vDash \varphi \to \psi & \Leftrightarrow & \mathsf{M}, w \nvDash \varphi \text{ or } \mathsf{M}, w \vDash \psi; \\
\mathsf{M}, w \vDash \Diamond_i \varphi & \Leftrightarrow & \exists v (w R_i v \ \& \ \mathsf{M}, v \vDash \varphi); \\
\mathsf{M}, w \vDash \Box_i \varphi & \Leftrightarrow & \forall v (w R_i v \Rightarrow \mathsf{M}, v \vDash \varphi).
\end{array}
$$

Ordered sums of frames: the polymodal case

$\mathsf{I} = (W, R)$ is a finite partial order, $W = \{w_1, \ldots, w_n\}$
$\mathsf{F}_1 = (W_1, R_1^1, \ldots, R_N^1), \ldots, \mathsf{F}_n = (W_n, R_1^n, \ldots, R_N^n)$ are N-frames;
$1 \le k \le N$. $\mathsf{I}[k; (\mathsf{F}_1, \ldots, \mathsf{F}_n)/(w_1, \ldots, w_n)]$:





For a class $\mathcal{F}$ of N-frames,

$$\sum_{\mathsf{I}:k} \mathcal{F} = \{\mathsf{I}[k; (\mathsf{F}_1, \ldots, \mathsf{F}_n)/(w_1, \ldots, w_n)] \mid \mathsf{F}_1, \ldots, \mathsf{F}_n \in \mathcal{F}\}.$$

For a class $\mathcal{I}$ of finite partial orders, put

$$\sum_{\mathcal{I}:k} \mathcal{F} = \bigcup \{\sum_{\mathsf{I}:k} \mathcal{F} \mid \mathsf{I} \in \mathcal{I}\}.$$

Lemma

Let $\mathcal{F}$ be a class of N-frames, $1 \leq k \leq N$. If an N-formula φ is $\sum_{\mathcal{PO}:k} \mathcal{F}$-satisfiable, then φ is $\sum_{\mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}:k} \mathcal{F}$-satisfiable.

An N-frame $\mathrm{G} = (W, R_1, \ldots, R_N)$ is called *rooted*, if for some $w \in W$ we have $\{w\} \cup R_1(w) \cup \cdots \cup R_N(w) = W$.

Lemma

Let $\mathcal{F}$ be a class of rooted N-frames closed under taking cones, $1 \leq k \leq N$. If $\mathcal{F}$-$\mathtt{Sat}$ is decidable in $O(n^d)$-space, then $\sum_{\mathcal{PO}:k} \mathcal{F}$-$\mathtt{Sat}$ is decidable in $O(n^{\max(3,d)})$-space.

For a class of N-frames $\mathcal{F}$, put:
$\mathcal{PO}^{(0)}[\mathcal{F}] := \mathcal{F}$;
$\mathcal{PO}^{(n+1)}[\mathcal{F}] := \sum_{\mathcal{PO}:1} \mathcal{G}$, where

$$\mathcal{G} = \{(W, \varnothing, R_1, \ldots, R_{n+N}) \mid (W, R_1, \ldots, R_{n+N}) \in \mathcal{PO}^{(n)}[\mathcal{F}]\};$$

$$\mathcal{PO}^{(\infty)}[\mathcal{F}] := \{\mathsf{F}_{(\varnothing)} \mid \mathsf{F} \in \mathcal{PO}^{(n)}[\mathcal{F}] \text{ for some } n\},$$

where $(W, R_1, \ldots, R_n)_{(\varnothing)}$ denotes $(W, R_1, \ldots, R_n, \varnothing, \varnothing, \ldots)$.

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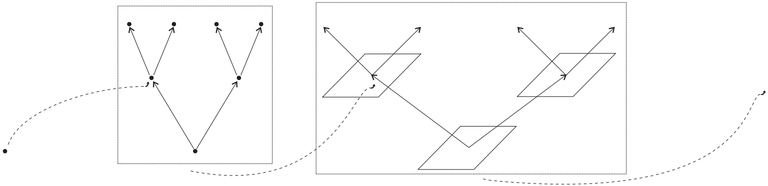
Theorem

Let $\mathcal{F}$ be a class of rooted N-frames closed under taking cones. If $\mathcal{F}$-$\mathcal{S}at$ is decidable in $O(n^d)$-space, then $\mathcal{PO}^{(\infty)}[\mathcal{F}]$-$\mathcal{S}at$ is decidable in $O(n^{\max(4,d)})$-space.

Stratified frames

Stratified frames: $\mathcal{PO}^{(\infty)}[\{C_0\}]$



Stratified frames

Stratified frames: $\mathcal{PO}^{(\infty)}[\{\mathsf{C}_0\}]$

Corollary

$\mathcal{PO}^{(\infty)}[\{\mathsf{C}_0\}]$-$\mathcal{S}at$ *is PSPACE-complete.*

Theorem (Beklemishev, 2007)

There exists a polynomial-time translation f such that for any formula φ we have

$$\mathrm{GLP} \vdash \varphi \Leftrightarrow \mathcal{PO}^{(\infty)}[\{\mathsf{C}_0\}] \vDash f(\varphi).$$

Corollary

Japaridze's Polymodal Logic is PSPACE-complete.