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Computability Theory

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Abstract. Computability is one of the fundamental notions of mathematics, trying to capture the effective content of mathematics. Starting from Gödel's Incompleteness Theorem, it has now blossomed into a rich area with strong connections with other areas of mathematical logic as well as algebra and theoretical computer science.

Mathematics Subject Classification (2000): 03Dxx, 68xx.

Introduction by the Organisers

The workshop *Computability Theory*, organized by Klaus Ambos-Spies and Wolfgang Merkle (Heidelberg), Steffen Lempp (Madison) and Rodney G. Downey (Wellington) was held February 5th–February 11th, 2012. This meeting was well attended, with 53 participants covering a broad geographic representation from five continents and a nice blend of researchers with various backgrounds in classical degree theory as well as algorithmic randomness, computable model theory and reverse mathematics, reaching into theoretical computer science, model theory and algebra, and proof theory, respectively. Several of the talks announced breakthroughs on long-standing open problems; others provided a great source of important open problems that will surely drive research for several years to come.

Workshop: Computability Theory

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Abstracts

Generic computability and asymptotic density

Carl Jockusch
(joint work with Rod Downey and Paul Schupp)

Often problems which are computationally difficult in principle can be easily solved in practice. For example, the simplex algorithm for linear programming has been shown to require exponential time, but runs very quickly on inputs which arise in practice. A theoretical model for this is "generic computability," which was introduced and studied by Kapovich, Miasnikov, Schupp and Shpilrain in [4] and applied to decision problems in group theory, among others. Generic computability was then studied in the context of classical computability theory by Jockusch and Schupp in [3], and further results in this area were obtained were obtained by Downey, Jockusch, and Schupp in [2]. Most of the results in this abstract are from the latter paper.

The basic definitions are as follows. Let $\omega = \{0, 1, \dots\}$. For $A \subseteq \omega$ and $n \in \omega$, let

$$\rho_n(A) = \frac{|\{k < n : k \in A\}|}{n}$$

be the density of A up to n. The *(asymptotic) density* $\rho(A)$ of A is defined as follows:

$$\rho(A) = \lim_n \rho_n(A)$$

provided the limit exists. We say that A is *generically computable* if there is a partial computable function φ such that φ agrees with the characteristic function of A on its domain, and its domain has density 1. Thus, there is a partial algorithm for A which never gives an incorrect answer, and answers with density 1. Further, define the *upper density* of A (denoted $\overline{\rho}(A)$) as $\limsup_n \rho_n(A)$, and the *lower density* of A (denoted $\underline{\rho}(A)$) as $\liminf_n \rho_n(A)$.

It is natural to ask whether the concept of generic computability remains unchanged if the requirement is added that the partial computable function φ has a computable domain. This is equivalent to asking:

(*) Does every c.e. set of density 1 have a computable subset of density 1?

This abstract is centered around this question and its variations.

It was shown in [3] that every c.e. set of upper density 1 has a computable subset of upper density 1. Further it was shown there that the number 1 can be replaced by any Δ^0_2 real.

For lower density, it was shown that for any c.e. set A and any real number $\epsilon > 0$, there exists a computable set $B \subseteq A$ such that $\underline{\rho}(B) \geq \underline{\rho}(A) - \epsilon$. Define A to have *effective* density 1 if $\rho_n(A)$ is computably convergent to 1. The method in the result just mentioned can be used to show that every c.e. set of effective density 1 has a computable subset of effective density 1. The argument can be combined with the limit lemma to show that every low c.e. set of density 1 has a computable subset of density 1.

In spite of these positive results, it was shown in [3] (Theorem 2.22), that there is a c.e. set of density 1 with no computable subset of density 1, and thus the answer to (*) above is "no". The proof of this is an infinite injury argument, but is remarkably simple because each requirement has only *one* opposing requirement, so the usual technical difficulties do not arise. This result was extended in [2] to show that there is a c.e. set A of density 1 such that no computable subset of A has nonzero density.

Once (*) is answered negatively, it is natural to ask about the degrees of the counterexamples. It is shown in [2] that the degrees of the c.e. sets A which have density 1 but no computable subsets of density 1 are exactly the *nonlow* c.e. degrees. One direction of this was already mentioned above. For the other direction, it must be shown that every nonlow c.e. set computes a c.e. set which has density 1 but no computable subset of density 1. This is done by introducing a new technique called "nonlow permitting" to the proof mentioned in the previous paragraph.

Call a set $B \subseteq \omega$ *absolutely undecidable* if every partial computable function which agrees with the characteristic function of B on its domain has domain of density 0 (or, equivalently, every c.e. subset of B or $\overline{B}$ has density 0). For example, every bi-immune set is absolutely undecidable. Obviously, every absolutely undecidable set is not generically computable, and it is easily seen that the converse fails. In our talk and in a draft of [2] we raised the question of whether every nonzero Turing degree contains an absolutely undecidable set. In [2] we obtained a partial result towards a negative answer by showing that there is a noncomputable set A such that for every set $B \leq_T A$, either B has a c.e. subset of positive upper density or $\overline{B}$ has an infinite c.e. subset. Nonetheless, Laurent Bienvenu, Adam R. Day, and Rupert Hölzl [1] have recently obtained a positive answer and in fact have given a fixed total truth-table reduction Φ such that Φ^A is absolutely undecidable whenever A is noncomputable.

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Are random axioms useful?

LAURENT BIENVENU
(joint work with Andrei Romashchenko, Alexander Shen, Antoine Taveneaux, and Stijn Vermeeren)

1. Introduction

The Kolmogorov complexity $C(x)$ of a binary string x is defined as the minimal length of a program (without input) that outputs x and terminates. This definition depends on a programming language, and one should choose one that makes complexity minimal up to $O(1)$ additive term. Most strings of length n have complexity close to n. More precisely, the fraction of n-bit strings that have complexity less than $n - c$, is at most 2^{-c}. In particular, there exist strings of arbitrary high complexity. (See [LV08, She00, DH10] for background information about Kolmogorov complexity and related topics.)

However, as G. Chaitin pointed in [Cha71], the situation changes if we look for strings of *provably* high complexity. More precisely, we are looking for strings x and numbers n such that the statement "$C(x) > n$" (for these x and n, so it is a closed statement) is provable in formal (Peano) arithmetic PA. Chaitin noted that all n that appear in these statements, are less than some constant c. Chaitin's argument is a version of the Berry paradox: Assume that for every integer k we can find some string x such that $C(x) > k$ is provable; let x_k be the first string in the order of enumeration of all proofs; this definition provides a program of size $O(\log k)$ that generates x_k, which is impossible for large k since $C(x_k) > k$.[1]

This leads to a natural idea. Toss a coin n times getting a string x of length n, and consider the statement $C(x) \geq n - 1000$. It is true unless we are extremely unlucky. The probability of being unlucky is less than 2^{-1000}. In natural sciences we are accustomed to identify this with impossibility. So we can add this statement and be sure that it is true; if n is large enough, we get a true non-provable statement and could use it as a new axiom. We can even repeat this procedure several times: if the number of iterations m is not astronomically large, $2^{-1000}m$ is still astronomically small.

Now the question: *Can we obtain a richer theory in this way and get some interesting consequences, still being practically sure that they are true*? The answers are given in Section 2 (using rather simple arguments):

- yes, this is a safe way of enriching our theory (PA), see Theorem 1;
- yes, we can get a stronger theory in this way (Chaitin's theorem), but
- no, we cannot prove anything interesting in this way, see Theorem 2.

[1] Another proof of the same result shows that Kolmogorov complexity is actually not very essential here: By a standard fixed-point argument one can construct a program p (without input) such that for every program q (without input) the assumption "q is equivalent to p" (i.e., q produces the same output as p if p terminates, and q does not terminate if p does not terminate) is consistent with PA. If p has length k, for every x we may assume without contradiction that p produces x, so one cannot prove that $C(x) > k$.

2. Probabilistic proofs in Peano arithmetic

2.1. **Random axioms: soundness.** Let us describe more precisely how we generate and use random axioms. Assume that some initial "capital" ε is fixed. Intuitively, ε measures the maximal probability that we agree to consider as "negligible".

The basic version: Let n and c be integers such that $2^{-c} < \varepsilon$. We choose at random (uniformly) a string x of length n, and add the statement $C(x) \geq n - c$ to PA (so it can be used together with usual axioms of PA).[2]

A slightly extended version: We fix several numbers $n_1, \ldots, n_k$ and $c_1, \ldots, c_k$ such that $2^{-c_1} + \ldots + 2^{-c_k} < \varepsilon$. Then we choose at random strings $x_1, \ldots, x_k$ of length $n_1, \ldots, n_k$, and add all the statements $C(x_i) \geq n - c_i$ for $i = 1, \ldots, k$.
Final version: In fact, we can allow even more flexible procedure of adding random axioms that does not mention Kolmogorov complexity explicitly. Assume that we already have proved for some number (numeral) N, for some rational $\delta > 0$ and for some property $R(x)$ (an arithmetical formula with one free string variable x) that *the number of strings x of length N such that $\neg R(x)$ does not exceed $\delta 2^N$* (the statement printed in italics is a closed arithmetical formula). Then we are allowed to toss a fair coin N times, generating a string r of length N, and add the formula $R(r)$ as a new axiom. This step can be repeated several times. We have to pay δ for each operation until the initial capital ε is exhausted; different operations may have different values of δ. (Our previous examples are special cases: the formula $R(x)$ says that $C(x) \geq n - c$ and $\delta = 2^{-c}$.) Note that the axiom added at some step can be used to prove the cardinality bound at next steps.[3]

In this setting we consider *proof strategies* instead of proofs. Such a proof strategy is a rooted tree whose internal nodes correspond to random choices made when a new axiom is added; note that the formula $R(x)$ used in the next step may depend on the random choice made at the previous step. The sum of "payments" δ along each path in the tree should not exceed ε. At each leaf of the tree we get some theory (PA plus the new axioms that are selected on the way from the root to this leaf). There is a natural probability distribution on leaves. Given a proof strategy π and a formula φ, we consider the probability that φ is provable by π. We do not assume here that a proof strategy is effective in any sense.

The following theorem says that this procedure indeed can be trusted:

Theorem 1 (soundness). *Let φ be some arithmetical statement. If the probability to prove φ for a proof strategy π with initial capital ε is greater than ε, then φ is true.*

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[2]As usual, we should agree on the representation of Kolmogorov complexity function C in PA. We assume that this representation is chosen in some natural way, so all the standard properties of Kolmogorov complexity are provable in PA. For example, one can prove in PA that the programming language used in the definition of C is universal. The correct choice is especially important when we speak about proof lenghts (Section 3).

[3]Actually this is not important: we can add previously added axioms as conditions.

2.2. **Random axioms are not useful.** As Chaitin's theorem shows, there are proof strategies that with high probability lead to *some* statements that are non-provable (in PA). However, the situation changes if we want to get some *fixed* statement, as the following theorem shows:

Theorem 2 (conservation). *Let φ be some arithmetical statement. If the probability to prove φ for a proof strategy π with initial capital ε is greater than ε, then φ is provable* (*in* PA *without any additional axioms*)*.*

Formally, Theorem 2 is a stronger version of Theorem 1, but the message here is quite different: Theorem 1 is the good news (probabilistic proof strategies are safe) whereas Theorem 2 is the bad news (probabilistic proof strategies are useless).

3. Polynomial size proofs

The situation changes drastically if we are interested in the length of proofs. The argument used in Theorem 2 gives an exponentially long "conventional" proof compared with the original "probabilistic" proof, since we need to combine the proofs for all terms in the disjunction. (Here the complexity of a probabilistic proof strategy is measured as the length of proof in the branch where this length is maximal; note that the total size of the proof strategy tree may be exponentially larger.) Can we find another construction that transforms probabilistic proof strategies into standard proofs with only polynomial increase in size? Probably not; some reason for this is provided by the following Theorem 3.

Theorem 3. *Assume that such a polynomially bounded transformation is possible. Then complexity classes* PSPACE *and* NP *coincide.*

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Cupping with Random Sets

Adam R. Day

(joint work with Joseph S. Miller)

Posner and Robinson proved that any non-computable set that is Turing below $\emptyset'$ can be cupped to $\emptyset'$ with a 1-generic set [9]. In 2004, Kučera asked which sets below $\emptyset'$ can be cupped to $\emptyset'$ with an incomplete Martin-Löf random [8]. In other words, does the Posner-Robinson theorem hold if we replace Baire category with Lebesgue measure, and if not, for which sets does it fail?

The basic definitions are the following. The prefix-free complexity of a string σ is denoted by $K(\sigma)$. A set R is *Martin-Löf random* if there exists some constant c such that for all n, $K(R \upharpoonright n) > n - c$. The definition of a Martin-Löf random set can be relativized to any oracle A. We call a set A, *low for Martin-Löf randomness* if every Martin-Löf random set is also Martin-Löf random relative to A. A set A is *K-trivial* if for some constant c, for all n we have that $K(A \upharpoonright n) \leq K(n) + c$ (where $K(n)$ is defined to be $K(1^n)$). Hence, a K-trivial set is indistinguishable from a computable set in terms of K complexity. (The existence of non-computable K-trivial sets was first established by Solovay in an unpublished manuscript [2]. Later, Zambella constructed a non-computable K-trivial c.e. set [12].) A set A is *weakly ML-cuppable* if $A \oplus X \geq_T \emptyset'$ for some incomplete Martin-Löf random set X. A is *ML-cuppable* if one can choose $X <_T \emptyset'$.

Kučera conjectured that the weakly ML-cuppable sets might be exactly the sets that are not K-trivial. Nies showed that there exists a non-computable K-trivial c.e. set that is not weakly ML-cuppable providing evidence for this conjecture [8]. We prove this conjecture showing that the K-trivial sets are precisely those sets that cannot be joined above $\emptyset'$ with an incomplete random. We also show that all sets below $\emptyset'$ that are not K-trivial, can be joined to $\emptyset'$ with a low random.

There are two directions to this proof. The first is that no K-trivial is weakly ML-cuppable. This direction uses the equivalence of the K-trivial sets and the low for Martin-Löf randomness sets, a result of Nies [7]. It also builds on work of Franklin and Ng, and Bienvenu, Hölzl, Miller and Nies. Franklin and Ng characterized the incomplete Martin-Löf random sets in terms of tests formed by taking the difference of two c.e. sets [3]. Recently, Bienvenu, Hölzl, Miller and Nies showed that the incomplete Martin-Löf random sets are exactly those Martin-Löf random sets for which a particular density property fails [1]. Our proof combines this density property with the fact that if A is a K-trivial set and W_A is a bounded set of strings c.e. in A, then there exists a bounded c.e. set of strings W such that $W_A \subseteq W$. (A set of finite strings S is *bounded* if $\sum_{\sigma \in S} 2^{-|\sigma|} < \infty$.) This fact was first explicitly stated, in an even stronger form, by Simpson [10]. It is also implied by work of Miller, Kjos-Hanssen and Solomon [4].

The second direction is that every set that is not K-trivial is ML-cuppable. This proof of this direction is an oracle construction using Π^0_1 classes of positive measure. Given A not K-trivial, we construct an R that is Martin-Löf random but not Martin-Löf random relative to A. From $A \oplus R$ we can determine the stage at which R leaves the n-th level of the universal Martin-Löf test relative to A. We construct R so that this stage is greater than the settling time for the first n bits of $\emptyset'$. This construction makes use of a result of Kučera that any Π^0_1 class of positive measure contains a tail of every Martin-Löf random [5].

Slaman and Steel extended the Posner-Robinson theorem to show that any non-computable set A that is strictly Turing below $\emptyset'$ can be cupped to $\emptyset'$ with a 1-generic set X such that A and X form a minimal pair [11]. The analogous result for A not K-trivial and X Martin-Löf random does not hold. Any Martin-Löf random computes a diagonally non-computable function. Kučera showed that if A and B both compute diagonally non-computable functions and are both below $\emptyset'$, then A and B do not form a minimal pair [6]. Hence no Martin-Löf random set below $\emptyset'$ forms a minimal pair with any set A below $\emptyset'$ that computes a diagonally non-computable function. However, it is unknown whether the following holds: for any non-computable set $A <_T \emptyset'$ that is not K-trivial, there is a Martin-Löf random $R <_T \emptyset'$ such that $A \oplus R \equiv_T \emptyset'$ and if $B \leq_T A, R$, then B is K-trivial.

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Problems of autostability and spectrum of autostability

Sergey Goncharov

In the study of autostability relative to strong constructivisations we consider the connection of autostability in almost prime models for complete decidable theories. It is additional results to my paper [1]. A. I. Maltsev gave start to systematic investigation of constructive models on the base of numberings or naming of elements of models and start to study algorithmic properties of structures on the base of classical theory of algorithms.

Let $\mathfrak{M}$ be a model with signature σ. If ν is an enumeration of the main set of model $\mathfrak{M}$, then we call the pair $(\mathfrak{M}, \nu)$ a numbered model. Here we take as value of every constant a_i an element $\nu(i)$ for each $i \in \mathbb{N}$. Let $D(\mathfrak{M}, \nu)$ be the quantifier-free theory of model $\mathfrak{M}_\nu$, i. e. the set of sentences without quantifier in signature $\sigma_\mathbb{N}$, which are true in model $\mathfrak{M}_\nu$.

The numbered model $(\mathfrak{M}, \nu)$ is constructive if the set $D(\mathfrak{M}, \nu)$ is recursive, i. e. there exists an algorithm for testing validity for quantifier-free formulas on elements of this model.

In connection with problem of uniqueness of constructive enumeration for a given model A. I. Maltsev introduced the notion of recursively stable model. He noticed that finitely generated algebraic systems are recursively stable.

Let $(\mathfrak{M}, \nu)$ and $(\mathfrak{M}, \mu)$ be two numbered models for the model $\mathfrak{M}$.

The numberings ν and μ of model $\mathfrak{M}$ are *recursively equivalent*, if there exist recursive functions f and g such that $\nu = \mu f$ and $\mu = \nu g$.

We can note that the two constructivizations ν and μ of model $\mathfrak{M}$ are *recursively equivalent*, if for any subsets $X \subseteq M^k, k \geq 1$ the set $\nu^{-1}(X)$ is recursive iff the set $\mu^{-1}(X)$ is recursive.

Let $Th(\mathfrak{M}, \nu)$ be the elementary theory of model $\mathfrak{M}_\nu$, i. e. the set of sentences in signature $\sigma_\mathbb{N}$, which are true in model $\mathfrak{M}_\nu$. A numbered model $(\mathfrak{M}, \nu)$ is called strongly constructive, if the elementary theory $Th(\mathfrak{M}, \nu)$ in signature $\sigma_\mathbb{N}$ is decidable.

Let Δ be a class of functions such that Δ is closed relative to superposition and for any permutation f of N from Δ the function f^{-1} from Δ too.

The constructivizations ν and μ of the model $\mathfrak{M}$ are Δ-*autoequivalent* relative to strong constructivization, if there exist function f from Δ and automorphism λ of the model $\mathfrak{M}$ such that $\lambda\nu = \mu f$.

The model is called Δ-*autostable* relative to strong constructivization if for every two strong constructivizations ν_1 and ν_2 of the model $\mathfrak{M}$ there exist automorphism α of model $\mathfrak{M}$ and function f from Δ such that $\alpha\nu_1 = \nu_2 f$.

We proved the main result, that for any $n \geq 5$ the existence of Ehrenfeucht theory with n countable models and the prime model of this theory is not autostable relative to strong constructivizations but some almost prime model of this theory is strongly constructivizable and autostable relative to strong constructivizations.

But it was proved in [1] that any decidable theory is not ω-stable if for this theory there exists a prime model in finite enrichment with constants such that it has a strong constructivization and is autostable relative to strong constructivizations but prime model is decidable and is not autostable relative to strong constructivizations.

Now we will consider an autostability relative to strong constructivizations in uncountably categorical theories down.

We proved that there can not exist an uncountably categorical theory with prime model, which is not autostable relative to strong constructivizations but with some other model, which is autostable relative to strong constructivizations. And if some model for uncountably categorical theory is autostable relative to strong constructivizations then its prime model will also be autostable relative to strong constructivizations.

Nevertheless for any n there exists decidable uncountably categorical theory such that all of its countable models are strongly constructivizable but only first n are autostable relative to strong constructivizations.

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Theories of Classes of Structures

Asher M. Kach
(joint work with Antonio Montalbán)

The authors started this project trying to answer a question from Ketonen [2]: Is the first-order theory of the class of countable Boolean algebras with the direct sum operation decidable? When he posed the question, Ketonen had recently answered the following question:

> Tarski's Cube Problem: Does there exist a countable Boolean algebra $\mathcal{B}$ such that $\mathcal{B} \cong \mathcal{B} \oplus \mathcal{B} \oplus \mathcal{B}$ but $\mathcal{B} \ncong \mathcal{B} \oplus \mathcal{B}$?

Indeed, Ketonen [3] shows that every countable commutative semi-group embeds in the commutative monoid of countable Boolean algebras, yielding a positive answer to Tarski's Cube Problem. In this talk, we discuss the answers to these questions when posed for other classes of structures: cardinals, vector spaces, equivalence structures, linear orders, groups, etc., as well as Boolean algebras.

We show that the first-order theory of the class of countable Boolean algebras with the direct sum operation is far from decidable; it is as complex as it can be.

Theorem 1. *The first-order theory of the class of countable Boolean algebras under the direct sum operation is* 1*-equivalent to true second-order arithmetic.*

For linear orders, we obtain a stronger result. For groups, we obtain a weaker result (we require the presence of the subgroup relation).

Theorem 2. *The class of countable linear orderings with the concatenation operation is bi-interpretable with second-order arithmetic.*

The first-order theory of the class of countable groups under the direct product operation and the subgroup relation is 1*-equivalent to true second-order arithmetic.*

To break the pattern, and to contrast with these results, we also give examples of theories which are decidable. These examples follow from work in Feferman and Vaught [1].

Corollary (Feferman and Vaught [1])**.** The theories of the following structures are decidable under ZFC.

- For any cardinal κ, the class of cardinals below κ under cardinal addition.

- For any countable field F, the class of countable F-vector spaces under direct sum.
- The class of countable equivalence structures under disjoint union.
- The class of finitely generated abelian groups under direct sum.

Surprisingly, this type of investigation of the theories of classes of algebraic structures seems to be in its infancy. Indeed, the only example in the literature the authors are knowledgeable about is the Ketonen [3] result already mentioned. As many natural questions remain open, the authors hope to see similar investigations in the future.

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Π^0_1 equivalence structures and their isomorphisms

Valentina Harizanov
(joint work with Douglas Cenzer and Jeffrey B. Remmel)

Equivalence relations play an important role in mathematical logic and many other areas of mathematics. For example, isomorphism and elementary equivalence, as well as their effective versions such as computable isomorphism or Σ^0_n equivalence, are equivalence relations. Similarly, a number of interesting applications of equivalence relations arise from the so-called classification problems where two structures are equivalent if they possess certain invariant properties. Here, we restrict our attention to countable structures for computable languages. If a structure is infinite, we can assume that its universe is the set of natural numbers, ω. Such a structure is computable if its atomic diagram is computable. Thus, an equivalence structure $\mathcal{A} = (\omega, E)$ is *computable* if E is computable.

A computable structure $\mathcal{A}$ is Δ^0_n *categorical* if every computable structure that is isomorphic to $\mathcal{A}$ is Δ^0_n isomorphic to $\mathcal{A}$. Thus, computable categoricity is the same as Δ^0_1 categoricity. A computable structure $\mathcal{A}$ is *relatively* Δ^0_n *categorical* if for every $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism that is Δ^0_α relative to the atomic diagram of $\mathcal{B}$. Clearly, relatively Δ^0_n categorical structures are Δ^0_n categorical. It can be shown that the converse may not be true. The study of the complexity of isomorphisms between computable equivalence structures was carried out by Calvert, Cenzer, Harizanov, and Morozov in [1]. Similarly, the study of equivalence structures within the Ershov difference hierarchy has been carried out by Cenzer, LaForte, and Remmel in [2].

We will now focus on Π^0_1 equivalence structures and their isomorphisms (see [3]). We say that an equivalence structure $\mathcal{A} = (\omega, E)$ is Π^0_1 (or *co-c.e.*) if E is a Π^0_1 set. Co-c.e. structures have been studied since the beginning of modern computable model theory. For example, in [5], Remmel studied co-c.e. structures where the underlying operations are computable.

The *character* of an equivalence structure $\mathcal{A}$ is the set

$$\chi(\mathcal{A}) = \{(k,n) : n,k > 0 \text{ and } \mathcal{A} \text{ has at least } n \text{ equivalence classes of size } k\}.$$

We say a character $\chi(\mathcal{A})$ is *bounded* if there is some finite k_0 such that for all $(k,n) \in \chi(\mathcal{A})$, we have $k < k_0$. By $Inf^{\mathcal{A}}$ we denote the set of all elements of $\mathcal{A}$ with infinite equivalence classes, and by $Fin^{\mathcal{A}}$ the set of all elements with finite equivalence classes.

We proved in [1] that a computable equivalence structure $\mathcal{A}$ is computably categorical if and only if $\mathcal{A}$ has either only finitely many finite equivalence classes, or $\mathcal{A}$ has finitely many infinite equivalence classes, bounded character, and exactly one finite k such that there are infinitely many equivalence classes of size k. We showed that computably categorical equivalence structures are relatively computably categorical. We further proved that a computable equivalence structure $\mathcal{A}$ is relatively Δ^0_2 categorical if and only if $\mathcal{A}$ has finitely many infinite equivalence classes or bounded character. We also established that all computable equivalence structures are relatively Δ^0_3 categorical.

We will now establish that even simple Π^0_1 equivalence structures do not have to be Δ^0_2 isomorphic to computable structures. First, we state the following positive result.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic Π^0_1 equivalence structures such that $\mathcal{A}$ satisfies one of the following conditions:*

- **(i):** *$\mathcal{A}$ has only finitely many finite equivalence classes, or*
- **(ii):** *$\mathcal{A}$ has finitely many infinite equivalence classes and bounded character, and there is at most one finite k such that $\mathcal{A}$ has infinitely many equivalence classes of size k.*

Then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.

Theorem 1 does not extend to other Π^0_1 equivalence structures isomorphic to relatively Δ^0_2 categorical structures.

Theorem 2. *Suppose that $\mathcal{B}$ is a computable equivalence structure that is relatively Δ^0_2 categorical, but not computably categorical. Then there exists an isomorphic Π^0_1 structure $\mathcal{A}$ that is not Δ^0_2 isomorphic to $\mathcal{B}$ and, moreover, $\mathcal{A}$ is not Δ^0_2 isomorphic to any Σ^0_1 structure.*

We note that Theorem 2 does not cover all Δ^0_2 categorical equivalence structures since Kach and Turetsky [4] showed that there exists a computable, Δ^0_2 categorical equivalence structure $\mathcal{B}$, which has infinitely many infinite equivalence classes and an unbounded character, and has only finitely many equivalence classes of size k for any finite k. The next result will cover this case.

Theorem 3. *Let $\mathcal{B}$ be a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character such that for each finite k, there are only finitely many equivalence classes of size k. Then there is a Π^0_1*

structure $\mathcal{A}$ that is isomorphic to $\mathcal{B}$ such that $Inf^{\mathcal{A}}$ is Π^0_2 complete. Furthermore, if $\mathcal{B}$ is Δ^0_2 categorical, then $\mathcal{A}$ is not Δ^0_2 isomorphic to any computable structure.

These results lead to the following general theorem.

Theorem 4. *Suppose that $\mathcal{B}$ is a computable equivalence structure that is not computably categorical. Then there is a Π^0_1 structure $\mathcal{A}$ that is isomorphic to $\mathcal{B}$, but is not Δ^0_2 isomorphic to $\mathcal{B}$.*

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Low level nondefinability in the Turing degrees

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(joint work with Mingzhong Cai)

A major focus of recent research on the Turing degrees, $\mathcal{D}$, has been definability and complexity. We discuss a topic that has received almost no attention but certainly deserves some: nondefinability. Now, outright nondefinability results would settle the major open question about the degrees by showing that they are not biinterpretable with second order arithmetic. We propose proving that sets of degrees, or relations on $\mathcal{D}$, are not definable by simple formulas in terms of quantifier complexity in various languages possibly stronger than the basic on with just $\leq$ (for Turing reducible to).

This work was motivated by the suggestion of Miller and Martin [5] that one could prove that the sets of hyperimmune (**HI**) and hyperimmune free (**HIF**) degrees are not simply definable. (A degree $\mathbf{x} \in$ **HIF** if and only if every $f \leq_T X$ is dominated by a recursive function. The class **HI** consists of the degrees not in **HIF**.) In addition to the basic language ($\leq$), they suggest two extensions augmenting it by either the jump operator $'$ or the relation RE for "recursively enumerable in." They also allow parameters $\bar{\mathbf{c}}$ for any specific degrees in all their languages. They prove that **HI** is not definable by a quantifier free formula in the language $(\leq, \bar{\mathbf{c}})$ (for any degrees $\bar{\mathbf{c}}$) and conjecture that this is also true for the language with jump,$(\leq,', \bar{\mathbf{c}})$. We affirm their conjecture and prove other similar theorems about these and other sets of, and relations on, degrees.

In particular, we are interested in another complementary pair of sets of degrees related to domination properties: the array nonrecursive degrees, **ANR**, are those computing a function not dominated by the modulus function for $0'$ and the complementary class of array recursive degrees, **AR**. We also suggest that in place of the relation RE one study REA, "recursively enumerable in and above." We note that while REA is clearly definable from RE by a quantifier free formula, we show that the reverse is not true even by a one quantifier formula (without parameters). In addition, our intuitions about relativizing the relation RE really apply to REA and all the examples we know using RE actually only use REA. Most striking among these is the two quantifier definition in this language (without parameters) of **ANR** as $\{\mathbf{x}|(\forall \mathbf{y} \geq \mathbf{x})(\exists \mathbf{z} < \mathbf{y})(\mathbf{y} \text{ is REA in } \mathbf{z}\}$ ([1]; [2]). There are also properties of REA degrees that we exploit in our constructions, in particular the characterization of such degrees in terms of the modulus lemma.

While some of our results allow parameters, most do not. Including parameters makes many more classes of degrees easily definable. Classic examples include the degrees of the productive sets: $\{\mathbf{x}|\mathbf{x} \geq \mathbf{0}'\}$; the immune sets: $\{\mathbf{x}|\mathbf{x} \geq \mathbf{0}\}$ and the functions dominating every recursive function: $\{\mathbf{x}|\mathbf{x}' \geq \mathbf{0}''\}$. However, unbridled use of parameters makes everything definable as by Slaman and Woodin, the biinterpretability conjecture holds for $\mathcal{D}$ with parameters (see [7, Corollary 5.6]) and so every relation on $\mathcal{D}$ definable in second order arithmetic is definable from parameters (although their proof does not produce any simple definitions).

The table below summarizes our current results listing classes and relations not definable by formulas in specific classes in one of the three languages indicated with or without parameters ($\bar{\mathbf{c}}$).

Class/Relation	Not defined by formula in	Language
HI	Σ_0	$(\leq, ', \bar{\mathbf{c}})$
HI	Σ_1	$(\leq, \bar{\mathbf{c}})$
HI	Σ_2	$(\leq)$
ANR	Σ_0	$(\leq, ', \bar{\mathbf{c}})$
ANR	Σ_1	$(\leq, \bar{\mathbf{c}})$
ANR	Σ_1, Π_1	$(\leq, REA)$
ANR	Σ_2	$(\leq)$
HIF	Σ_1	$(\leq, REA, \bar{\mathbf{c}})$
HIF	Σ_2	$(\leq)$
AR	Σ_1	$(\leq, REA, \bar{\mathbf{c}})$
Jump	Σ_1	$(\leq, REA)$
RE	Σ_1, Π_1	$(\leq, REA)$

We note the particularly tight instances. **ANR** is definable by a two quantifier formula in $(\leq, REA)$ but not by any one quantifier one. It is also not definable by a Σ_2 formula in just $(\leq)$. The relation $\mathbf{y} = \mathbf{x}'$ is definable in $(\leq, REA)$ by a Π_1 formula $(\forall z(zREAx \to z \leq y)$ but not by a Σ_1 formula in this language.

All the analyses proceed by first finding an appropriate syntactic normal form. Typically a second step is to simplify the form by considering degrees in the class being considered with special properties such as avoiding cones determined by the parameters, being minimal or perhaps also r.e. in some other degree (or not). Then one needs to do a construction to show that whatever diagram provided the witnesses that the special degrees are in the class can be duplicated with corresponding degrees outside it. This then shows that the formula cannot define the class being considered. The techniques for the two quantifier results with just $\leq$ depend on standard initial segment and extension of embedding results and follow similar ones for other classes in Lerman and Shore [4] and Shore [6]. The ones not involving REA are at the moment mostly ad hoc exploiting various forcing constructions and special facts.

Most all the results involving REA depend on variations on a basic theorem providing a decision procedure for one quantifier formulas in the language $(\leq, REA)$.

Definition 1. A partial order $\leq$ is an *REA partial order* if it has an additional binary relation $x\ REA\ y$ such that $x\ REA\ y \to x \geq y$ and $x\ REA\ y\ \&\ x \geq z \geq y \to x\ REA\ z$.

Theorem 2. *Every finite REA partial order can be embedded in* $(\mathcal{D}, \leq, REA)$.

The proof is a forcing argument which enforces $x\ REA\ y$ by explicitly putting a function $l(n,s)$ limit computing X into both X and Y and a modulus function $m(n)$ for this computation into X. Of course, enforcing the negative relations is the combinatorially complicated part of the argument. The entries in the table above involving REA are now proven by variations on this construction that make at least some of the degrees realizing a particular diagram be in specific classes. For example, while the basic construction uses two genericity, more careful arguments produce superlow degrees and hence ones in both **HI** and **AR**. Examples of simplifying the diagram part of the arguments include, for example for the relation RE, the existence of a minimal degree r.e. in $\mathbf{0}'$ but not recursive in it.

Now it is obvious that there are many natural open questions raised by this work. Perhaps the most obvious ones are to include parameters in the results that do not now have them and to replace REA by RE. Even adding a parameter for $\mathbf{0}$ seems a challenging but probably accessible problem using priority arguments in place of forcing. Controlling the relation RE rather than REA calls for a different kind of coding and forcing argument (even without parameters). Of course, moving up to two quantifier sentences for languages stronger than the basic one is an obvious but seemingly difficult problem. Even with just $\leq$, the nondefinability of **AR** by a Σ_2 formula should require some new idea. At the one quantifier level, the natural questions would include extending the language by adding join and considering jump as well. Some of these seem quite tractable.

Ted Slaman suggested the degrees $\mathcal{A}$ of the arithmetic sets as a good candidate for proving a sharp definability result as we already have a fairly low level definition in the language with just $\leq$. In fact, by rewriting [3, Theorem 3.3] to eliminate join, the class of arithmetic degrees is definable by a Σ_3 formula in just $\leq$: $\mathcal{A} = \{\mathbf{x} | (\exists \mathbf{y} \geq \mathbf{x})(\forall \mathbf{z} \forall \mathbf{w}(\mathbf{w} \geq \mathbf{z}, \mathbf{y} \to \exists \mathbf{u}(\mathbf{w} > \mathbf{u} > \mathbf{z}))\}$. On the other hand, the arguments of [6, Proposition 7.6] also show that $\mathcal{A}$ is not definable by either a Σ_2 or Π_2 formula and so we have as simple a definition as possible. The same arguments (in both directions) work for the relation of being arithmetic in. It too is $\Sigma_3(\leq)$ but neither Σ_2 nor Π_2. Note that as for any countable ideal, $\mathcal{A}$ is quantifier free $(\leq, \bar{\mathbf{c}})$ definable (from an exact pair for the ideal). It is easy to see that the relation "arithmetic in" is not so definable.

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Balancing Randomness

Dan Turetsky

(joint work with Laurent Bienvenu, Noam Greenberg, Antonín Kučera, and André Nies)

In [2], Kučera gave an injury-free solution for Post's problem, by showing that every Δ^0_2 Martin-Löf random set computes a noncomputable c.e. set. In hindsight, this result initiated a broad research programme, which lies at the intersection of algorithmic randomness and the study of the computably enumerable (c.e.) Turing degrees. In general, researchers stud the distribution of the random sets in the Turing degrees, and in particular how these random degrees fit in with other classes of degrees which are examined by classical computability theory, prime among them being the class of c.e. degrees. Since incomplete c.e. sets cannot compute random sets, the natural question to ask is: which random sets compute which c.e. sets?

The K-trivial sets are a natural class which arise repeatedly in the study of algorithmic randomness. For example, they are precisely the sets which cannot derandomize any Martin-Löf random. Hirschfeldt, Nies and Stephan showed in [1] that every c.e. set which is computable from an incomplete Martin-Löf random set is K-trivial. The converse is known as the covering problem, and it remains open.

Kučera and Slaman showed in [3] that there is a low set which computes every K-trivial set, although their low set was not Martin-Löf random. A natural strengthening of the covering problem is then, "Is there a low Martin-Löf random which computes every K-trivial set?" We answer this question in the negative, while also providing evidence against the covering problem.

We obtain our results by examining a new form of randomness, which we are tentatively calling Oberwolfach randomness.

Definition 1. An *Oberwolfach test* is a pair $(\langle G_\sigma \rangle_{\sigma \in 2^{<\omega}}, \alpha)$, where $\alpha < 1$ is a left c.e. real, and $\langle G_\sigma \rangle_{\sigma \in 2^{<\omega}}$ is a uniformly Σ^0_1 sequence such that $\lambda G_\sigma \leq 2^{-|\sigma|}$.

A real X *passes* an Oberwolfach test $(\langle G_\sigma \rangle_{\sigma \in 2^{<\omega}}, \alpha)$ if $X \notin \bigcap_n G_{\alpha \upharpoonright n}$.

A real is *Oberwolfach random* if it passes every Oberwolfach test.

Oberwolfach randomness turns out to be "near" Martin-Löf randomness, in the sense that a Martin-Löf random real must be fairly powerful as an oracle in order to avoid being Oberwolfach random.

Theorem 2. *If Z is Martin-Löf random but not Oberwolfach random, then Z is h-JT-hard for any computable order h with $\sum_x \frac{1}{h(x)} < \infty$, and hence Z is super-high.*

So Oberwolfach randomness includes a large amount of the incomplete Martin-Löf randoms. It may in fact contain all of them; we are yet unable to answer this question. We are, however, able to understand it's relationship to K-triviality.

Theorem 3. *There is a c.e. K-trivial set which is not computable from any Oberwolfach random.*

Putting this together with the previous result, we see that this K-trivial is only computable from Martin-Löf randoms which are super-high.

Theorem 4. *If Z is Martin-Löf random but not Oberwolfach random, then Z computes every K-trivial set.*

So the Oberwolfach randoms are precisely the Martin-Löf randoms which do not compute every K-trivial set. The covering problem can thus be rephrased as "Is there an incomplete Oberwolfach random?"

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The Typical Turing Degree

Andrew E. M. Lewis
(joint work with George Barmpalias and Adam Day)

I'll describe a research project, which is joint work with George Barmpalias and Adam Day, and which addresses the issue, "what properties are satisfied by the typical Turing degree?"

The inspiration for this line of research begins essentially with Kolmogorov's 0-1 law, which states that any (Lebesgue) measurable tailset is either of measure 0 or 1. Since Turing degrees are tailsets, this means that upon considering properties which may or may not be satisfied by any given Turing degree, we can immediately conclude that, so long as the satisfying class is measurable, it must either be of measure 0 or 1. Thus either the *typical degree* satisfies the property, or else the typical degree satisfies its negation, and this suggests an obvious line of research. Initially we might concentrate on definable properties, where by a definable set of Turing degrees we mean a set which is definable as a subset of the structure in the language of partial orders. For each such property we can look to establish whether the typical degree satisfies the property, or whether it satisfies the negation. In fact we can do a little better than this. If a set is of measure 1, then there is some level of algorithmic randomness which suffices to ensure membership of the set. So, once we have established that the typical degree satisfies a certain property, we may also look to establish the level of randomness required in order to ensure typicality as far as the given property is concerned.

Lebesgue measure though, is not the only way in which we can gauge typicality. One may also think in terms of Baire category. For each definable property, we may ask whether or not the satisfying class is comeager and, just as in the case for measure, it is possible to talk in terms of a hierarchy which allows us to specify levels of typicality. The role that was played by randomness in the context of measure, is now played by genericity. For any given comeager set, we can look to establish the level of genericity which is required to ensure typicality in this regard.

During our research, we have isolated the following heuristic principle: *if a property holds for all highly random/generic degrees then it is likely to hold for all non-zero degrees that are bounded by a highly random/generic degree.* Here by 'highly random/generic' we mean at least 2-random/generic. Thus, establishing levels of typicality which suffice to ensure satisfaction of a given property, also gives a way of producing lower cones and sets of degrees which are downward closed (at least amongst the non-zero degrees), such that all of the degrees in the set satisfy the given property.

Over the course of a number of months we have systematically considered various degree theoretic-properties, starting with the most simple and working our way on to properties which are more complex. We have developed frameworks which seem to apply very widely, and which we have used to give a good number of new results. Our research in this area is written up in two papers, 'Measure and cupping in the Turing degrees' and 'The typical Turing degree'. The first of these papers is joint with George and the second is joint with both George and Adam.

I'll list some of the new results here, and refer you to the papers for a discussion which places these results properly in the context of the previous literature.

While it was known that the measure of the degrees which bound a minimal degree is 0, the level of randomness required was left open as a question in the Downey, Hirschfeldt book. We answered this by proving:

Theorem 1. *Every non-zero degree bounded by a 2-random bounds a 1-generic.*

One can then show that this is sharp. It is not too hard to show that there exist Demuth random degrees which bound minimal degrees. The following result suffices to show that there are weakly 2-random degrees which bound minimal degrees (since a 1-random real is weakly 2-random if and only if its degree forms a minimal pair with $\mathbf{0}'$, and every generalized high degree bounds a minimal degree).

Theorem 2. *Given a Π^0_1 class P of positive measure, there is $A \in P$ which is generalized high, and whose degree forms a minimal pair with $\mathbf{0}'$.*

This also gives a rather simple answer to a question of Nies, as to whether all weakly 2-random sets are array computable. We were also able to answer a question of Jockusch by showing:

Theorem 3. *The measure of the degrees which satisfy the cupping property is 0. In fact, every degree below a 2-random degree has a strong minimal cover.*

On the other hand, we can show that no degree below a 2-random degree is a strong minimal cover. Combined with results by Jockusch, Theorem 3 means that every 2-random degree and every 2-generic degree form a minimal pair.

Moving on to the join property, we were able to strengthen the result of Jockusch that every 2-generic degree satisfies join:

Theorem 4. *Every 1-generic degree satisfies the join property.*

Also, the measure of the degrees which satisfy the join property is 1:

Theorem 5. *Every degree bounded by a 2-random degree satisfies the join property.*

While we have not yet been able to determine whether the meet and complementation properties are satisfied by all sufficiently random degrees, we have the following partial result:

Theorem 6. *Every non-zero degree bounded by a 2-random is the top of a diamond.*

For those those who are interested, a detailed account of the previous literature is given in 'The typical Turing degree', but in case you want to go straight to the previous literature, some of the most useful references are [9, 11, 10, 1, 3, 4, 5, 6, 7, 8, 2].

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Maximal chains of computable well partial orders

Alberto Marcone

(joint work with Antonio Montalbán and Richard Shore)

The proofs of the results stated in this abstract will appear in [4].

If P is partially ordered by $\leq_P$, $C \subseteq P$ is a chain in P if the restriction of $\leq_P$ to C is linear. If P is a well-founded partial order then every chain in P is a well-order and we define the height of P, $\mathrm{ht}(P)$, to be the supremum of all ordinals which are order types of chains in P. For $x \in P$, let $\mathrm{ht}_P(x)$ be the supremum of all ordinals which are order types of chains in $P_{(-\infty,x)} = \{\, y \in P \mid y <_P x \,\}$. It is easy to see that $\mathrm{ht}(P) = \sup\{\, \mathrm{ht}_P(x) + 1 \mid x \in P \,\}$ and that $\mathrm{ht}_P(x) = \sup\{\, \mathrm{ht}_P(y) + 1 \mid y <_P x \,\}$.

Definition 1. Let C be a chain in the well-founded partial order P:

- C is *maximal* if it has order type $\mathrm{ht}(P)$;
- C is *strongly maximal* if, for every $\alpha < \mathrm{ht}(P)$, there exists a (necessarily unique) $x \in C$ with $\mathrm{ht}_P(x) = \alpha$.

Recall that a well partial order (from now on wpo) is a well-founded partial order with no infinite antichains.

Wolk proved the following theorem, which appears also in Harzheim's book ([3, Theorem 8.1.7]):

Theorem 2 ([10]). *Every wpo has a strongly maximal chain.*

We can now state precisely the questions we are interested in:

Question 1. If P is a computable wpo, how complicated must maximal and strongly maximal chains in P be?

It follows from previous work in [5] that a computable wpo always has a hyperarithmetic strongly maximal chain.

Question 2. How complicated must any function taking the wpo P to such a maximal chain be?

As usual, the computability of P means that both $P \subseteq \mathbb{N}$ and $\leq_P \subseteq \mathbb{N} \times \mathbb{N}$ are (Turing) computable sets. In our answers to these questions, we will measure complexity in terms of Turing computability as well as the hyperarithmetic hierarchy which is built by iterating the Turing jump (halting problem) along computable well orderings.

Theorem 2 is somewhat similar to the better known result of de Jongh and Parikh:

Theorem 3 ([2])**.** *Every wpo P has a maximal linear extension, i.e. there exists a linear extension of P such that every linear extension of P embeds into it. We call such a linear extension a maximal linear extension.*

Montalbán answered the analogues of Questions 1 and 2 for maximal linear extensions.

Theorem 4 ([6])**.** *Every computable wpo has a computable maximal linear extension, yet there is no hyperarithmetic way of computing (an index for) a computable maximal linear extension from (an index for) the computable wpo.*

Our first result concerns strongly maximal chains in computable wpos and shows that they can be of arbitrarily high complexity in the hyperarithmetical hierarchy.

Theorem 5. *Let $\alpha < \omega_1^{\mathrm{CK}}$. There exists a computable wpo P such that any strongly maximal chain in P computes $0^{(\alpha)}$.*

Our second result shows that maximal chains can also be highly noncomputable. In contrast to Theorem 5, however, we do not show that they must lie arbitrarily high up in the hyperarithmetic hierarchy.

Theorem 6. *Let $\alpha < \omega_1^{\mathrm{CK}}$. There exists a computable wpo P such that $0^{(\alpha)}$ does not compute any maximal chain in P.*

Theorem 6 cannot be strengthened by constructing a computable wpo P such that any maximal chain in P computes $0^{(\alpha)}$. In fact, we show that maximal chains in wpos can be computed from (Cohen) generic sets (defined e.g. in [7, §IV.3]):

Theorem 7. *If P is a computable wpo and G a hyperarithmetically generic set then $C \leq_T G$ for some maximal chain C in P. Furthermore, if P has a maximal chain of length $< \omega^{\alpha+1}$, then $2 \cdot \alpha$-genericity of G suffices.*

Theorem 7 implies that almost every set, in the sense of category, computes maximal chains, and that every computable wpo has a maximal chain that does not compute any noncomputable hyperarithmetic set.

Our proof of Theorem 7 is nonuniform, and this cannot be avoided:

Theorem 8. *There is no hyperarithmetic procedure which calculates a maximal chain in every computable wpo.*

In fact, suppose f is such that, for every index e for a computable wpo P, $n \mapsto f(e,n)$ is (the characteristic function of) a maximal chain of P. Then f computes every hyperarithmetic set.

Our results illustrate several interesting differences between the analysis of complexity in terms of computability strength as done here and axiomatic strength in the sense of reverse mathematics as is done in [5]. (See [9] for basic background in reverse mathematics.) From the viewpoint of reverse mathematics, all of the theorems analyzed computationally here and in [6] are equivalent. Indeed, in [5] the first and third author showed that Theorem 3 and Theorem 2 (indeed even the version for maximal chains) for countable wpos are each equivalent (over RCA_0) to the same standard axiom system, ATR_0. As we have explained, however, the computational analysis of these three theorems in the sense of Question 1 are quite different.

Computable partial orders all have computable maximal linear extensions by Theorem 4. Computable wpos all have hyperarithmetic maximal and even strongly maximal chains as is shown by the proof in ATR_0 of Theorem 2 in [5]. However, strongly maximal chains for computable wpos must be of arbitrarily high complexity relative to the hyperarithmetic sets while maximal chains can be computably incomparable with all noncomputable hyperarithmetic sets. Yet another level of computational complexity within the theorems axiomatically equivalent to ATR_0, is provided by König's duality theorem (every bipartite graph G has a matching M such that there is a cover of G consisting of one vertex from each edge in M). (See [1] for definitions.) Here [1] and [8] show that this theorem for countable graphs is equivalent to ATR_0. On the other hand, [1, Theorem 4.12] shows that there is a single computable graph G such that any cover as required by the theorem already computes every hyperarithmetic set and so this G certainly has no such hyperarithmetic cover.

The phenomena exhibited by our analysis of the existence of maximal chains seems to be new.

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Algorithmic Randomness in Ergodic Theory

Henry Towsner
(joint work with Johanna Franklin)

A *dynamical system* consists of a set of points Ω with a σ-algebra $\mathcal{B}$ of subsets of Ω and a probability measure μ on $\mathcal{B}$, together with a measureable, measure-preserving function $T : \Omega \to \Omega$.

One of the first theorems proven about such systems is the Birkhoff Ergodic Theorem,

Theorem 1 (Birkhoff)**.** *For any measurable set A, the average*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i<N} \chi_A(T^i x)$$

converges for almost every x.

The average in this theorem is known as the *ergodic average*.

From the perspective of algorithmic randomness, it is natural to restrict A to some countable collection with some type of computability requirement and say that a point x is random exactly if the average converges for every such set A and for every transformation T in some class. (Here, and throughout, it is possible to generalize χ_A to an L^1 function without changing the results.)

T is *ergodic* exactly if the average almost always converges to $\mu(A)$. The case where T is ergodic has received a fair amount of attention recently, almost completely resolving the question:

Theorem 2.

(1) If x is Schnorr random, T is computable and ergodic, and A is computably enumerable with computable measure then the ergodic average of x converges. [1, 6]

(2) If x is not Schnorr random then there is a computable ergodic T and a computable set A such that the ergodic average of x fails to converge. [6]

(3) If x is Martin-Löf random, T is computable and ergodic, and A is computably enumerable then the ergodic average of x converges. [2, 5]

(4) *If x is not Martin-Löf random then there is a computable ergodic T and a computably enumerable set A such that the ergodic average of x does not converge to $\mu(A)$.* [2]

It appears to be open whether this last result can be improved so that x fails to converge at all.

The non-ergodic case has not been as well studied. The only previous result we are aware of is:

Theorem 3 ([8]). *If x is Martin-Löf random, T is computable, and A is computably enumerable with computable measure then the ergodic average of x converges.*

By using the method of *cutting and stacking*, a standard method of ergodic theory for constructing transformations with particular properties (see [4, 7]), we are able to show the converse:

Theorem 4. *If x is not Martin-Löf random then there is a computable T and a computable set A such that the ergodic average of x fails to converge*

We turn to the question of non-ergodic transformations with computably enumerable sets. The proof of Theorem 3 depends on the notion of an *upcrossing*:

Definition 5. Given a set A and bounds $\alpha < \beta$, $\nu_A(x)$ is defined to be the supremum over N such that there exist $u_1 < v_1 < u_2 < v_2 < \cdots < u_N < v_N$ such that for each $k \leq N$,

$$\frac{1}{u_k}\sum_{i<u_k}\chi_A(T^ix) < \alpha < \beta < \frac{1}{v_k}\sum_{i<v_k}\chi_A(T^ix).$$

Note that $\nu_A(x)$ is infinite if and only if the ergodic average at x fails to converge.

Bishop's constructivization of the ergodic theorem shows

Theorem 6 ([3]).

$$\int \nu_A(x)d\mu \leq \mu(A)\frac{1-\alpha}{\beta-\alpha}.$$

In order to extend this argument to computably enumerable sets, we must consider the question of how stable the set of points with many upcrossings is under small changes to A. We introduce a general notion of upcrossing which leads to a rather weak bound.

Definition 7. Given sets $A \subseteq B$ and bounds $\alpha < \beta$, $\tau_{A,B}(x)$ is defined to be the supremum over N such that there exist $u_1 < v_1 < u_2 < v_2 < \cdots < u_N < v_N$ such that for each $k \leq N$,

$$\frac{1}{u_k}\sum_{i<u_k}\chi_A(T^ix) < \alpha < \beta < \frac{1}{v_k}\sum_{i<v_k}\chi_B(T^ix).$$

Theorem 8. *If $\mu(B \setminus A) < \epsilon$, there is a set W with $\mu(W) < \frac{4\epsilon}{\beta-\alpha}$ such that*

$$\int_{\Omega\setminus W} \tau_{A,B}(x) d\mu$$

is finite.

This implies:

Theorem 9. *If x is weakly 2-random, T is computable, and A is computably enumerable then the ergodic average of x converges.*

For a potential converse, we do know of any result stronger than Theorem 4 even when A is allowed to be computably enumerable.[1]

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Proving that Artinian implies Noetherian without proving that Artinian implies finite length

Chris Conidis

Let R be a commutative ring with identity. Recall from basic graduate algebra that:

1. R is Noetherian if it satisfies the ascending chain condition on its ideals;
2. R is Artinian if it satisfies the descending chain condition on its ideals; and
3. R is of finite length if there is a uniform bound on the length of any (strictly increasing/decreasing) chain of ideals in R.

It is well-known that 2. implies 1., but the proofs given in most standard algebra texts prove this by showing the stronger statement that 2. implies 3. This begs the question: "Can one prove that 2. implies 1. without showing that 2. implies 3?

[1]Since the Oberwolfach conference, a suggestion of Kohlenbach's appears to have lead to a uniformization of Theorem 8, which in turns leads us to believe that Theorem 9 can be improved to hold whenever x is *Oberwolfach-random*, a notion announced by Turetsky at the same conference.

We will show that this is indeed the case by showing that, in the context of reverse mathematics, the former (weaker) statement is equivalent to weak König's lemma, while the latter (stronger) statement is equivalent to arithmetic comprehension in the context of ω-models. Another way to view our main result is that it constructs an ω-model of RCA (recursive comprehension axiom) in which 2. implies 1., but 2. does not imply 3.

The proof of our main result is based on the fact that annihilators play an influential role in the theory of Artinian rings. In other words, the main difference between our new proofs and the standard old proofs is that, instead of considering general ideals in Artinian rings, we almost always restrict ourselves to ideals that are annihilators or finite intersections of annihilators.

Copy or diagonalize

Antonio Montalbán

We introduce notions that describe computability aspects of classes of structures. This definitions are new, but they reflect ideas that have been around for a few decades.

The first notions are based in the following game. Let $\mathbb{K}$ be a nice class of structures. The game is played by two players, C and D (for copy and diagonalize), and they alternatively play finite structures $\mathcal{C}[0], \mathcal{D}[0], \mathcal{C}[1], \mathcal{D}[1], ...$ such that $\mathcal{C}[i] \subseteq \mathcal{C}[i+1]$ and $\mathcal{D}[i] \subseteq \mathcal{D}[i+1]$ for all i. Let $\mathcal{C} = \bigcup_{i\in\omega} \mathcal{C}[i]$ and $\mathcal{D} = \bigcup_{i\in\omega} \mathcal{D}[i]$. If $\mathcal{C}$ and $\mathcal{D}$ are both in $\mathbb{K}$, then C wins is $\mathcal{C} \cong \mathcal{D}$ and D wins if $\mathcal{C} \not\cong \mathcal{D}$. If $\mathcal{C} \notin \mathbb{K}$, then D wins, and if $\mathcal{C} \in \mathbb{K}$ but $\mathcal{D} \notin \mathbb{K}$, then D wins. Finally, we add an extra rule, that if a player wants to play a finite structure, he has to eventually play a special symbol '$\square$' to mark that he is done constructing.

We say that $\mathbb{K}$ is *copyable* if C has a winning strategy in this game, and we say that $\mathbb{K}$ is *diagonalizable* otherwise.

We also define a version of this game where C builds infinitely many structures $\mathcal{C}^i$ and he has to have one of them isomorphic to $\mathcal{D}$; hence D needs to diagonalize against all $\mathcal{C}^i$ simultaneously. Our main theorem is the following

Theorem 1. *Let $\mathbb{K}$ be Π^c_2-axiomatizable with a computable 1-back-and-forth structure. The following are equivalent:*

- $\mathbb{K}$ *has the low property.*
- $\mathbb{K}'$ *is listable.*
- $\mathbb{K}'$ *is ∞-copyable.*

The $\mathcal{D}$-maximal sets

Peter Cholak
(joint work with Peter Gerdes and Karen Lange)

We discuss the classification of the $\mathcal{D}$-maximal sets.

Definition 1 (The sets disjoint from A)**.** $\mathcal{D}(A) = \{B : \exists W(B \subseteq A \cup W \text{ and } W \cap A =^* \emptyset)\}$ under inclusion. Let $\mathcal{E}_{\mathcal{D}(A)}$ be $\mathcal{E}$ modulo $\mathcal{D}(A)$.

Definition 2. A is *$\mathcal{D}$-maximal* iff $\mathcal{E}_{\mathcal{D}(A)}$ is the trivial Boolean algebra iff for all c.e. sets B there is a c.e. set D disjoint from A such that either $B \subset A \cup D$ or $B \cup D \cup A = \omega$.

Maximal sets, hemimaximal sets, Herrmann sets and sets with A-special lists are $\mathcal{D}$-maximal.

We show that there are many more $\mathcal{D}$-maximal sets and classify how they are generated. In fact the class of $\mathcal{D}$-maximal sets breaks up into infinitely many orbits. For all but finitely many of these orbits, it is unknown if these orbits contain complete sets.

Connectedness and the Brouwer Fixed Point Theorem

Vasco Brattka
(joint work with Stéphane le Roux and Arno Pauly)

We study the computational content of the Brouwer Fixed Point Theorem in the Weihrauch lattice. One of our main results is that for any fixed dimension the Brouwer Fixed Point Theorem of that dimension is computably equivalent to connected choice of the Euclidean unit cube of the same dimension. Connected choice is the operation that finds a point in a non-empty connected closed set given by negative information. Another main result is that connected choice is complete for dimension greater or equal to three in the sense that it is computably equivalent to Weak Kőnig's Lemma. In contrast to this, the connected choice operations in dimensions zero, one and two form a strictly increasing sequence of Weihrauch degrees, where connected choice of dimension one is known to be equivalent to the Intermediate Value Theorem. Whether connected choice of dimension two is strictly below connected choice of dimension three or equivalent to it is unknown, but we conjecture that the reduction is strict. As a side result we also prove that finding a connectedness component in a closed subset of the Euclidean unit cube of any dimension greater or equal to one is equivalent to Weak Kőnig's Lemma. In order to describe all these results we introduce a representation of closed subsets of the unit cube by trees of rational complexes.

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The Stable Ramsey's Theorem for Pairs

Theodore A. Slaman
(joint work with Chong Chi Tat and Yang Yue)

We investigate the properties of Ramsey's Theorem for Pairs in the context of subsystems of second order arithmetic.

Definition 1. For $X \subseteq \omega$, let $[X]^n$ denote the size n subsets of X. For $n, m > 0$ and $F : [\omega]^n \to \{0, \dots, m-1\}$, $H \subseteq \omega$ is *homogeneous for F* iff F is constant on $[H]^n$.

Theorem 2 (Ramsey, 1930, [2])**.** *For all $n, m > 0$ and all $F : [\omega]^n \to \{0, \dots, m-1\}$, there is an infinite set H such that H is homogeneous for F.*

If we fix n and m, then we represent that instance of Ramsey's Theorem by RT^n_m.

Definition 3. A *model* $\mathfrak{M}$ *of second-order arithmetic* consists of a structure $\mathfrak{N}$ for first-order arithmetic, called the *numbers* of $\mathfrak{M}$, and a collection of subsets of $\mathfrak{N}$, called the *reals* of $\mathfrak{M}$.

Definition 4. RCA_0 is the second-order theory formalizing the following.

- P^-, the axioms for the nonnegative part of a discretely ordered ring.
- $I\Sigma_1$, for ϕ a Σ^0_1 predicate, if 0 is a solution to ϕ and the solutions to ϕ are closed under successor, then ϕ holds of all numbers.
- The reals are closed under join and relative Δ^0_1-definability.

Definition 5. In an *ω-model* $\mathfrak{M}$, $\mathfrak{N} = \omega$ and the reals of $\mathfrak{M}$ form an ideal in the Turing degrees.

We can decompose of RT^2_2 over RCA_0 into the two principles COH and SRT^2_2, defined as follows.

Definition 6.
- An infinite set X is *cohesive* for a family $R_0, R_1, \ldots$ of sets iff for each i, one of $X \cap R_i$ or $X \cap \overline{R_i}$ is finite. COH is the principle stating that every family of sets has a cohesive set.
- A partition $F : [\omega]^2 \to \omega$ is *stable* iff for all x, $\lim_{y\to\infty} F(x,y)$ exists. SRT^2_2 is the principle RT^2_2 restricted to stable partitions.

These two principles together comprise RT^2_2.

Theorem 7 (Cholak, Jockusch, and Slaman, 2001, [1])**.** $RCA_0 \vdash \left[RT^2_2 \iff (SRT^2_2 \,\&\, COH)\right]$

We settle a question from [1] about whether this decomposition is proper by showing that SRT^2_2 does not imply RT^2_2.

Theorem 8 (Chong, Slaman, and Yang)**.** *There is a model $\mathfrak{M}$ of RCA_0 with the following properties.*
- $\mathfrak{M} \models SRT^2_2$
- $\mathfrak{M} \models \neg I\Sigma_2$
- *Every real in $\mathfrak{M}$ is low in $\mathfrak{M}$.*

However, we have been unable to determine whether every ω-model of $RCA_0 + SRT^2_2$ is also a model of RT^2_2, which we regard as an extremely interesting question.

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Amalgamation constructions and recursive model theory

Uri Andrews

We explore the use of Hrushovski amalgamation constructions to answer problems of computable model theory. Hrushovski originally created these constructions to answer a geometric question of model theory from Zilber: Must every strongly minimal theory have a geometry arising from an algebraic structure? More specifically, Zilber conjectured that every strongly minimal theory is locally modular or field-like. The Hrushovski construction provides examples of strongly minimal theories which are neither, and in fact these have inherently combinatorial rather than algebraic nature.

There are two natural reasons that one might expect Hrushovski constructions to lend themselves to yielding computationally interesting examples. Firstly, combinatorial structures are easier to use to code recursion theoretic information than algebraic ones. Secondly, the more structure an object respects, the easier it is to compute information about the structure. Thus, if a strongly minimal theory also respects geometric structure, it is more likely to yield to easier computation.

Using Hrushovski amalgamation constructions, we provide examples of strongly minimal theories answering open questions in computable model theory. In particular, we show the following theorems:

Theorem 1 (A., [An2]). *Let* $\mathbf{d}$ *be any tt-degree* $\leq 0^\omega$. *Then there exists both strongly minimal and* $\aleph_0$*-categorical theories with finite signatures in* $\mathbf{d}$ *all of whose countable models are recursively presentable.*

Theorem 2 (A., [An1][An3]).

- *There exists strongly minimal theories* T_n *with finite signatures so that only the models of dimension* $0, 1, \dots n$ *admit recursive presentations.*
- *There exists a strongly minimal theory with finite signature so that only the models of finite dimension admit recursive presentations.*
- *There exists a strongly minimal theory with finite signature so that only the saturated model admits a recursive presentation.*
- *There exists a strongly minimal theory with finite signature so that only the prime and saturated models admit recursive presentations.*

Each of the results in Theorem 2 were achieved by use of a Hrushovski amalgamation construction. This naturally leads to the question of whether this is necessary. In other words, could there be a strongly minimal theory with finite signature satisfying Zilber's conjecture with any of these spectra? The following presents some results, joint with Alice Medvedev, where we begin to hint at the answer being 'no'. We conjecture, in fact, that such spectra are in fact very limited:

Conjecture 3 (A.-Medvedev, [AnMe]). If T is a strongly minimal theory in a finite language satisfying the Zilber conjecture, then the collection of recursively presentable models of T is $\emptyset$, $\{$all countable models of $T\}$, or $\{$the prime model of $T\}$.

Some evidence for the conjecture comes from the following:

Theorem 4 (A.-Medvedev, [AnMe]). *If* T *is a disintegrated strongly minimal theory in a finite language, then the collection of recursively presentable models of* T *is* $\emptyset$, $\{$*all countable models of* $T\}$, *or* $\{$*the prime model of* $T\}$.

As all of the known recursive spectra of theories (before Theorem 2) were constructed with disintegrated theories, this shows that *some* new approach was necessary. The below results, by proving the conjecture for the canonical examples of theories satisfying the Zilber conjecture, begin to hint that such a new approach must also be a counterexample to Zilber's conjecture.

Theorem 5 (A.-Medvedev, [AnMe]). *If T is a locally modular theory in a finite language which expands a group, then the collection of recursively presentable models of T is $\emptyset$, $\{$all countable models of $T\}$, or $\{$the prime model of $T\}$.*

Theorem 6 (Poizat, [Po88]). *If T is a field-like theory in a finite language which expands a field, then every countable model of T admits a recursive (in fact decidable) presentation. (In fact, this is a corollary of Poizat's result that any field-like strongly minimal expansion of a field is a definitional expansion)*

Thus we see that Hrushovski amalgamation constructions can be used as a new tool to construct interesting examples in recursive model theory, and in fact, we are beginning to see the necessity of this new approach.

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Mathias generic sets

Damir D. Dzhafarov
(joint work with Peter A. Cholak and Jeffry L. Hirst)

We present some results about generics for *computable Mathias forcing*, in which conditions are pairs (D, E) such that D is a finite set, E is an infinite computable set, and $\max D < \min E$, and a condition (D', E') extends (D, E) if $D \subseteq D' \subseteq D \cup E$ and $E' \subseteq E$. This forcing notion has been a prominent tool for constructing infinite homogeneous sets for computable colorings of pairs of integers, as in Seetapun and Slaman [6], Cholak, Jockusch, and Slaman [2], and Dzhafarov and Jockusch [3]. It has also found applications in algorithmic randomness, in Binns, Kjos-Hanssen, Lerman, and Solomon [1].

For $n \geq 3$, we call a set G *Mathias n-generic* if it meets or avoids every Σ^0_n collection of conditions, and *weakly n-generic* if it meets every dense such collection. The n-generic sets and weakly n-generic sets in this setting form a strict hierarchy as in the case of Cohen forcing. Many other results concerning Cohen generics hold also for Mathias generics, but a number do not. The main point of distinction is that neither the set of conditions nor the relation of extension are computable, so many usual techniques do not carry over.

We begin with an analysis of the Mathias forcing relation, which differs from that of Cohen forcing past Σ^0_2 statements.

Lemma 1. *Let (D, E) be a condition and let $\varphi(X)$ be a formula in exactly one free set variable.*

(1) If φ is Σ^0_0 then the relation $(D, E) \Vdash \varphi(G)$ is computable.
(2) If φ is Π^0_1, Σ^0_1, or Σ^0_2, then so is the relation $(D, E) \Vdash \varphi(G)$.
(3) For $n \geq 2$, if φ is Π^0_n then the relation of $(D, E) \Vdash \varphi(G)$ is Π^0_{n+1}.
(4) For $n \geq 3$, if φ is Σ^0_n then the relation $(D, E) \Vdash \varphi(G)$ is Σ^0_{n+1}.

Among other results, this allows us to prove a strengthening of the well-known fact that every sufficiently Mathias generic is high.

Theorem 2. *If G is n-generic then it has $\mathbf{GH}_1$ degree, i.e., $G' \equiv_T (G \oplus \emptyset')'$.*

It follows that no Cohen 1-generic set can compute a Mathias n-generic set, since for all $m \geq 1$, every Cohen m-generic satisfies $G^{(m)} \equiv_T G \oplus \emptyset^{(m)}$, as shown by Jockusch [4, Lemma 2.6]. We have the following analogue of this result for Mathias genericity:

Theorem 3. *For all $n \geq 2$, if G is n-generic then $G^{(n-1)} \equiv_T G' \oplus \emptyset^{(n)}$.*

Jockusch and Posner [5, Corollary 7] showed that every $\overline{\mathbf{GL}}_2$ degree bounds a Cohen 1-generic degree. Thus, another consequence of Theorem 2 is that every Mathias n-generic set computes a Cohen 1-generic. This leads to the following question, which we have so far been unable to answer.

Question 1. Does every Mathias n-generic set compute a Cohen n-generic set?

We conjecture the answer to the question to be no. However, the following theorem establishes that Mathias n-generics do not require much additional computational power to compute Cohen n-generics. Recall that a set is *co-immune* if its complement has no infinite computable subset.

Theorem 4. *If G is Mathias n-generic, and $A \leq_T \emptyset^{(n-1)}$ is bi-immune (i.e., A and $\overline{A}$ are each co-immune), then $G \oplus A$ computes a Cohen n-generic.*

Proof. Let $\mathcal{C}_0, \mathcal{C}_1, \ldots$ be a listing of all Σ^0_n subsets of $2^{<\omega}$, together with fixed $\emptyset^{(n-1)}$-computable enumerations. For each i, let $\mathcal{D}_i$ be the set of all conditions (D, E) such that $D \cap A$, viewed as a binary string of length $\min E$, belongs to $\mathcal{C}_i$. Then $\mathcal{D}_i$ is a Σ^0_n set of conditions, and as such must be met or avoided by G. If G meets $\mathcal{D}_i$ then $G \cap A$, viewed as an element of 2^ω, meets $\mathcal{C}_i$. If G avoids $\mathcal{D}_i$, we claim that $G \cap A$ must avoid $\mathcal{C}_i$. Indeed, suppose G avoids $\mathcal{D}_i$ via (D, E). Since A and $\overline{A}$ are co-immune, they intersect E infinitely often, and so if $D \cap B$ had an extension τ in $\mathcal{C}_i$, we could make a finite extension (D', E') of (D, E) so that $D' \cap A = \tau$. This extension would belong to $\mathcal{D}_i$, a contradiction. □

It follows, for example, that the join of G with any non-computable $S \leq_T \emptyset'$ computes a Cohen n-generic.

The reason for using co-immune sets above is because the infinite part of a Mathias condition can be made arbitrarily sparse along any computable domain, and so any computable set of coding locations can always be removed. Our last result shows that this method is, in some sense, the only way of coding information into Mathias conditions. Hence, the proof of the above theorem cannot be used also for $S = \emptyset$.

Proposition 5. *If G is Mathias n-generic and H is Cohen n-generic then H is not many-one reducible to G.*

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Isolation: Motivations and Applications

Guohua Wu
(joint work with Mars Yamaleev)

In this talk, we will describe how the isolation phenomenon can be obtained from Kaddah's work in 1993, and then show interactions between isolation and other structural properties, like nonbounding, cupping, and bubble constructions.

Say that a $(n+1)$-c.e. degree $\mathbf{d}$ is isolated by an n-c.e. degree $\mathbf{c} < \mathbf{d}$, if there is no $(n+1)$-c.e. degree between $\mathbf{c}$ and $\mathbf{d}$. The notion of isolation was proposed by Cooper and Yi in [3] for $n = 1$ and by LaForte in [6] for $n > 1$. The isolation phenomenon has its origins in Kaddah's thesis (see [5]):

Theorem 1. *(Kaddah [5])*

1. *Every low c.e. degree $\mathbf{c}$ is the infimum of two d.c.e. degrees $\mathbf{d}$ and $\mathbf{e}$. (In case that $\mathbf{c}$ is nonbranching in the c.e. degrees, then one of the intervals $(\mathbf{c}, \mathbf{d})$ and $(\mathbf{c}, \mathbf{e})$ contains no c.e. degrees, and hence $\mathbf{c}$ isolates either $\mathbf{d}$ or $\mathbf{e}$.)*
2. *For $n \geq 2$, there are two n-c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that they have infimum $\mathbf{d}$ in the n-c.e. degrees, and there also exists an $(n+1)$-c.e. degree $\mathbf{x}$ such that $\mathbf{d} < \mathbf{x} < \mathbf{a}, \mathbf{b}$. (Note that there are no n-c.e. degrees between $\mathbf{d}$ and $\mathbf{x}$, so $\mathbf{x}$ is isolated by $\mathbf{d}$ in the n-c.e. degrees.)*

Early work of Ding and Qian (96), LaForte (96), and Arslanov, Lempp and Shore (96) showed that the isolated d.c.e. degrees and the nonisolated d.c.e. degrees are dense in the c.e. degrees. In contrast to this, Salts proved in 2000 that the nonisolating degrees are not dense in the c.e. degrees. Ishmukhametov and Wu proved in 2003 that the isolated d.c.e. degrees can be far away from the associated isolating c.e. degree, in terms of the high/low hierarchy.

The isolation phenomenon also appears in Arslanov, Kalimullin, Lempp's recent work, showing that $\mathbf{D}_3$, but not $\mathbf{D}_2$, contains 3-bubbles - this provides a proof that $\mathcal{D}_2$ and $\mathcal{D}_3$ are not elementarily equivalent.

Theorem 2. *(Arslanov, Kalimullin and Lempp* [1]*) There exist a d.c.e. degree* $\mathbf{d}$ *and a c.e. degree* $\mathbf{c}$*, such that* $\mathbf{0} < \mathbf{c} < \mathbf{d}$ *and any d.c.e. degree* $\mathbf{u} \leq \mathbf{d}$ *is comparable with* $\mathbf{c}$*.*

By Sacks splitting theorem (avoid cones), there is no c.e. degree between $\mathbf{c}$ and $\mathbf{d}$, and hence $\mathbf{d}$ is isolated by $\mathbf{c}$.

Our recent work of using isolation provides a strong cupping theorem, which unifies several well-known results in the d.c.e. degrees, such as Arslanov's cupping theorem, Downey's diamond theorem and Li-Yi's cupping theorem.

Say that a d.c.e. degree $\mathbf{d}$ has almost universal cupping property if it cups every c.e. degree not below it to $\mathbf{0}'$. Obviously, the incomplete maximal d.c.e. degrees constructed by Cooper, et al. in [2] do have this property. However, constructing a d.c.e. degree with almost universal cupping property is much easier than constructing an incomplete maximal d.c.e. degree. This enables us to prove the following theorem:

Theorem 3. *(Fang, Liu and Wu* [4]*) For any nonzero cappable c.e. degree* $\mathbf{c}$*, there is a d.c.e. degree* $\mathbf{d}$ *with almost universal cupping property and a c.e. degree* $\mathbf{b} < \mathbf{d}$ *such that* $\mathbf{b}$ *isolates* $\mathbf{d}$ *and that* $\mathbf{c}$ *and* $\mathbf{b}$ *form a minimal pair.*

Applying this theorem twice will give a proof of Li-Yi's cupping theorem: there exist two incomparable d.c.e. degrees $\mathbf{d}$ and $\mathbf{e}$ such that any nonzero d.c.e. degree cups one of them to $\mathbf{0}'$.

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Properties of limitwise monotonicity spectra of $\mathbf{\Sigma^0_2}$ sets

Iskander Kalimullin
(joint work with M.Kh. Faizrahmanov)

The talk is devoted to limitwise monotonic spectra. Such spectra can be considered as a partial case of spectra of uniform enumerability of families of computable sets.

Definition. The talk is devoted to measure and category properties of limitwise monotonicity spectra of Σ^0_2 sets. Such spectra can be considered as a partial case of spectra of uniform enumerability of families of computable sets.

- A countable family $\mathcal{F} \subset 2^\omega$ is (uniformly) $\mathbf{x}$-*c.e.* if $\mathcal{F} = \{W^X_{f(n)} : n \in \omega\}$ for some computable function f and $X \in \mathbf{x}$.
- The *enumeration spectra* of a family $\mathcal{F}$ is the collection of Turing degrees defined as
$$\mathbf{SpE}\,(\mathcal{F}) = \{\mathbf{x} : \mathcal{F} \text{ is } \mathbf{x}\text{-c.e.}\}.$$
- Given an infinite set S. The set S is $\mathbf{x}$-*limitwise monotonic* if the family
$$\mathcal{LM}(S) = \{\omega \upharpoonright m : m \in S\}$$
is $\mathbf{x}$-c.e.
- The *limitwise monotonicity spectra* is the collection $\mathbf{SpE}\,(\mathcal{LM}(S))$.

Note that the limitwise monotonicity spectra of a set S coincides with the degree spectrum of the abelian p-group $\sum_{m\in S} \mathbb{Z}_{p^m}$.

Theorem. (Kalimullin, Faizrahmanov). *If $\mathcal{S}$ is uniformly Σ^0_2 then*

(1) $\mathbf{GH}_1 \subseteq \mathbf{SpE}\,(\mathcal{LM}(\mathcal{S}))$*;*
(2) $\mathbf{2\text{-}generic} \subseteq \mathbf{SpE}\,(\mathcal{LM}(\mathcal{S}))$*;*
(3) $\mathbf{1\text{-}generic}\ \cap \Delta^0_2 \cap \mathbf{SpE}\,(\mathcal{LM}(\mathcal{S})) \neq \emptyset$*.*
(4) $\mathbf{1\text{-}random} \subseteq \mathbf{SpE}\,(\mathcal{LM}(\mathcal{S}) \implies \mathbf{0} \in \mathbf{SpE}\,(\mathcal{LM}(\mathcal{S}))$*;*

Theorem. (Kalimullin, Faizrahmanov). *There is an $\mathcal{S} \in \Delta^0_2$ such that $\mathbf{SpE}\,(\mathcal{LM}(\mathcal{S}))$ is null.*

Corollary. (Greenberg, Montalban, Slaman [1]). *There is a structure whose degree spectrum is co-meager and null.*

Moreover, the structure from this corollary can be an abelian p-group in the form $\sum_{m\in S} \mathbb{Z}_{p^m}$.

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On the number of K-trivial sequences

George Barmpalias
(joint work with Tom Sterkenburg)

Kolmogorov complexity is a standard tool for measuring the initial segment complexity of an infinite binary sequence. Identifying subsets of $\mathbb{N}$ with their characteristic sequence, we say that the initial segment (Kolmogorov) complexity of $A \subseteq \mathbb{N}$ is *trivial* if it is bounded by the Kolmogorov complexity of a computable set (modulo a constant). This concept was introduced by Chaitin (see [Cha76, Sol75]). For the case of plain Kolmogorov complexity Chaitin [Cha76] showed that any set with trivial initial segment complexity must be computable. In the case of the prefix-free complexity K, Solovay [Sol75] showed that there are non-computable sets with trivial initial segment complexity. In the last decade these so-called K-trivial sets, the sets A with $\forall n\ K(A\restriction_n) \leq K(n) + c$ for some $c \in \mathbb{N}$, have been the subject of intense research in algorithmic randomness. They are known to form a very interesting Σ^0_3 class $\mathcal{K}$ which is an ideal in the Turing degrees, see [Nie09, Chapter 5].

The members of $\mathcal{K}$ are stratified in a cumulative hierarchy where level e consists of the sets A such that $\forall n\ K(A\restriction_n) \leq K(n) + e$. In this case we say that A is K-trivial with constant e, or even that e is a K-triviality constant for A. Chaitin [Cha76] also showed that for each e there are finitely many K-trivial sets with constant e. A question of Downey/Miller/Yu (see [DH10, Section 10.1.4] and [Nie09, Problem 5.2.16]) asked about the complexity of the following problem.

(1) Given $e \in \mathbb{N}$, find the number of K-trivial sets with constant e.

Let $\mathcal{K}_e$ denote the class of K-trivial sets with constant e. The following question refers to the complexity of the function $e \to |\mathcal{K}_e|$.

Question 1 (Section 10.1.4 in [DH10] and Problem 5.2.16 in [Nie09])**.** What is the arithmetical complexity of (1)? In particular, is it Δ^0_3?

In this work we give a positive answer to the above question, thus showing that the function $e \to |\mathcal{K}_e|$ lies exactly at level Δ^0_3 of the arithmetical hierarchy. In particular, although the function $e \to |\mathcal{K}_e|$ depends on the choice of the underlying universal machine, its arithmetical complexity does not.

The solution of this problem gives a general methodology for answering the same question for related Σ^0_3 classes, like the finite K-trivial sets or the low for K sets. A set A is low for K if it does not compress strings more efficiently than a computable oracle. More precisely, if the prefix-free complexity K^A relative to A is not smaller than their unrelativized prefix-free complexity K, modulo a constant. In symbols, if $K(\tau) \leq K^A(\tau) + c$ for some constant c and all strings τ. This Σ^0_3 class can also be seen as the union of a cumulative hierarchy whose eth level consists of the sets A with $K(\tau) \leq K^A(\tau) + e$ for all strings σ. As in the case of $\mathcal{K}$ we say that A is low for K with constant e if it lies in the eth level of this hierarchy. Hirschfeldt and Nies showed in [Nie05] that the class of low for K sets coincides with $\mathcal{K}$. However this coincidence is not effective, in the sense that there is no algorithm which outputs a level in the low for K hierarchy where a set lives, given a level of it in the K-triviality hierarchy. Hence determining the complexity of the functions giving the cardinality of the levels of the two hierarchies constitutes two separate problems.

Further applications of our methodology concern the plain complexity versions of the above triviality notions.

The positive answer to Question 1 can also be used in order to refine the work of Csima and Montalbán in [CM06]. In this paper the authors construct an unbounded nondecreasing function f such that for all reals X and all constants c

$$\text{if } \forall n\ (K(X\restriction_n) \leq K(n) + f(n) + c) \text{ then } X \text{ is } K\text{-trivial.}$$

An analysis of the construction shows that f is Δ^0_4. The complexity of f can be reduced to Δ^0_3 using the answer to Question 1 as it is shown in [BB10]. Moreover this is optimal in the sense that if f is Δ^0_2 then it does not have the above property. This was shown in [BV11] and in [BB10] it was extended, showing that if f is Δ^0_2 unbounded and nondecreasing then there is a large and rich collection of reals X which satisfy $\forall n\ (K(X\restriction_n) \leq K(n) + f(n) + c)$ for some constant c.

The purpose of constructing f in [CM06] was the construction of minimal pairs with respect to $\leq_K$, where $A \leq_K B$ if $\exists c\, \forall n\ (K(A\restriction_n) \leq K(B\restriction_n) + c)$. According to the above discussion the complexity of minimal pairs is reduced to Δ^0_3. However different methods in [BV11] give a simpler construction of a Σ^0_2 set that forms a minimal pair with every Σ^0_1 with respect to $\leq_K$.

The main remaining open problem concerns the computational power of the function $e \to |\mathcal{K}_e|$. Let $\emptyset'$ be the halting problem and let $\emptyset''$ be the halting problem relativized to $\emptyset'$.

Problem. Does the function $e \to |\mathcal{K}_e|$ compute $\emptyset'$ or even $\emptyset''$?

Our methods do not seem sufficient for the solution of this problem. Furthermore, the answer is likely to depend on the choice of the underlying universal prefix-free machine.

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An example related to a theorem of John Gregory

Julia Knight

(joint work with Jesse Johnson, Victor Ocasio, and Steven VanDenDriessche)

Around 1970, parts of model theory (infinitary logic), set theory (fine structure of L), and computability (α-recursion) had very close ties. Jon Barwise proved a Compactness Theorem for infinitary logic, and he became a great spokesperson for the unity of logic. There were many contributors to the body of work, including Ronald Jensen, Carole Karp, Georg Kreisel, Saul Kripke, Richard Platek, John Schlipf, Jean-Pierre Ressayre, and Gerald Sacks and some of his students, in particular, Richard Shore and Sy Friedman. We state the Barwise Compactness Theorem [1], and then Gregory's Theorem [2].

Theorem 1 (Barwise Compactness Theorem)**.** *Let A be a countable admissible set and let T be a set of L_A sentences that is Σ_1 on A. If every A-finite subset of T has a model, then T has a model.*

For the case where A is the least admissible set containing ω, the L_A sentences are essentially the computable infinitary sentences, a set is Σ_1 on A iff it is Π^1_1, and a set is A-finite iff it is hyperarithmetical.

Theorem 2 (Gregory)**.** *Let A be a countable admissible set. Suppose T is a set of L_A-sentences that is Σ_1 on A. If T has a pair of countable models $\mathcal{M},\mathcal{N}$ s.t. $\mathcal{M} \prec_{L_A} \mathcal{N}$, then T has an uncountable model.*

Gregory's proof was quite clever. There is a simpler proof[3], using Ressayre's notion of Σ-saturation [4]. Gregory said that there were known examples showing that the assumption T is Σ_1 on A cannot be dropped. He did not give an example, and we have been unable to find one published.

There is current work on absoluteness of statements asserting the existence of an uncountable member of a "non-elementary" class K—K may be the class of models of a sentence of $L_{\omega_1\omega}$ or $L_{\omega_1\omega}(Q)$, or it may be an "abstract elementary class". John Baldwin is involved in joint work of this kind with Martin Körwein, Typani Hyttinen, and Sy Friedman, and also with Paul Larson. Baldwin asked for an example illustrating Gregory's Theorem. He believed (correctly) that the example would involve computability. We describe an example.

Theorem 3 (Johnson-K-Ocasio-VanDenDriessche)**.** *There is a set T of computable infinitary sentences, in a computable language L, s.t. T has just two models, $\mathcal{M}$ and $\mathcal{N}$, up to isomorphism, where $\mathcal{M},\mathcal{N}$ are countable and $\mathcal{M} \prec_\infty \mathcal{N}$.*

Moreover, for each computable ordinal α, the set of computable Σ_α sentences in T is hyperarithmetical.

The language of T consists of unary predicates U_n for $n \in \omega$. Each L-structure represents a family of sets. The set represented by an element x is the set of n s.t. $U_n x$ holds. The universe of $\mathcal{M}$ is an infinite computable set of constants C, partitioned effectively into infinitely many infinite sets C_n. The extra element of $\mathcal{N}$ is a further constant a. We identify the constants with the sets they represent, once we have determined these sets.

The set a will be "hyperarithmetically" generic; i.e., it is α-generic for all computable ordinals α. We choose an increasing sequence of computable ordinals $(\alpha_n)_{n\in\omega}$ with limit ω_1^{CK}. We suppose that $\alpha_n + \alpha_{n+1} = \alpha_{n+1}$. For all $c \in C_n$ and all $k < n$, $U_k c$ iff $U_k a$. Apart from this, the elements of C_n will be mutually α_n-generic, and uniformly $\Delta^0_{\alpha_n+1}$. We choose the set a in advance. We then use iterated forcing to choose the families of sets C_n, first C_0, then C_1, etc. We must prove the following.

Proposition 4.

A *$\mathcal{M}$ and $\mathcal{N}$ are the only models of T, up to isomorphism.*
B $\mathcal{M} \prec_\infty \mathcal{N}$

For **A**, it is enough to note that the computable infinitary theory of $\mathcal{M}$ and $\mathcal{N}$ includes the following.

(1) sentences saying that all elements that are not $\Delta^0_{\alpha_n+1}$ satisfy the same predicates U_k for $k \leq n$,
(2) sentences saying exactly which $\Delta^0_{\alpha_n+1}$ sets are represented,
(3) a sentence saying that distinct elements differ on some U_k.

For **B**, we must understand truth in the structures. The control that we have of truth in $\mathcal{M}$ and $\mathcal{N}$ comes from forcing. We do not decide truth in $\mathcal{M}$ or $\mathcal{N}$ directly. Let $\mathcal{M}_n$ be the structure with universe $\cup_{k\leq n} C_k$, and let $\mathcal{N}_n$ be the structure with universe $\cup_{k\leq n} C_k \cup \{a\}$. We choose a in advance, deciding truth in $\mathcal{N}_n$, for all possible choices of $\mathcal{M}_n$. When we choose C_n, having already chosen $C_{<n}$, we decide truth in $\mathcal{M}_n$. We prove three lemmas.

Lemma 5. $\mathcal{M}_n \prec_{\alpha_n} \mathcal{M}_{n+1}$. *(This implies that $\mathcal{M}_n \prec_{\alpha_n} \mathcal{M}$.)*

Lemma 6. $\mathcal{M}_n \prec_{\alpha_n} \mathcal{N}_n$

Lemma 7. $\mathcal{N}_n \prec_{\alpha_n} \mathcal{N}_{n+1}$. *(This implies that $\mathcal{N}_n \prec_{\alpha_n} \mathcal{N}$.)*

Assuming the three lemmas, we finish as follows.

Proposition B. $\mathcal{M} \prec_\infty \mathcal{N}$

Proof. Suppose $\mathcal{N} \models \varphi(\overline{c})$, where $\varphi(\overline{c})$ is computable Σ_{α_n} and $\overline{c}$ is in $\mathcal{M}_n$. Since $\mathcal{M}_n \prec_{\alpha_n} \mathcal{N}_n \prec_{\alpha_n} \mathcal{N}$, $\varphi(\overline{c})$ holds in $\mathcal{N}_n$ and in $\mathcal{M}_n$. Since $\mathcal{M}_n \prec_{\alpha_n} \mathcal{M}$, it also holds in $\mathcal{M}$.

□

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Computable Differential Fields

Russell Miller
(joint work with Alexey Ovchinnikov and Dmitry Trushin)

Differential algebra is the study of differential equations from a purely algebraic standpoint. The differential equations studied are polynomials in a variable Y and its derivatives $\delta Y, \delta(\delta Y), \ldots$, with coefficients from a specific field K which admits differentiation on its own elements via the operator δ. Such a field K is known as a *differential field*: it is simply a field with one or more additional unary functions δ on its elements, satisfying the usual properties of derivatives: $\delta(x + y) = (\delta x) + (\delta y)$ and $\delta(x \cdot y) = (x \cdot \delta y) + (y \cdot \delta x)$. It is therefore natural to think of the field elements as functions, and standard examples include the field $\mathbb{Q}(X)$ of rational functions in one variable under differentiation $\frac{d}{dX}$, and the field $\mathbb{Q}(t, \delta t, \delta^2 t, \ldots)$ with a *differential transcendental* t satisfying no differential equation over the ground field $\mathbb{Q}$. Additionally, every field becomes a differential field when the operator $\delta x = 0$ is adjoined; we call such a differential field a *constant field*, since an element whose derivative is 0 is commonly called a *constant*.

Although the natural examples are fields of functions, the treatment of differential fields regards the field elements merely as points. There are strong connections between differential algebra and algebraic geometry, with such notions as the ring $K\{Y\}$ of differential polynomials (namely the algebraic polynomial ring $K[Y, \delta Y, \delta^2 Y, \ldots]$, with each $\delta^i Y$ treated as a separate variable), differential ideal, differential variety, and differential Galois group all being direct adaptations of the corresponding notions from field theory. Characteristically, these concepts behave similarly in both areas, but the differential versions are often a bit more complicated. In terms of model theory, the theories $\mathbf{ACF}_0$ and $\mathbf{DCF}_0$ (of algebraically closed fields and differentially closed fields, respectively, of characteristic 0) are both complete and ω-stable with effective quantifier elimination, but $\mathbf{ACF}_0$ has Morley rank 1, whereas $\mathbf{DCF}_0$ has Morley rank ω.

Just as the algebraic closure $\overline{F}$ of a field F (of characteristic 0) can be defined as the prime model of the theory $\mathbf{ACF}_0 \cup \Delta(F)$ (where $\Delta(F)$ is the atomic diagram of F), the differential closure $\hat{K}$ of a differential field K is normally taken to be the prime model of $\mathbf{DCF}_0 \cup \Delta(K)$. This $\hat{K}$ is unique up to isomorphism over K, but not always minimal: it is possible for $\hat{K}$ to embed into itself over K (i.e.

fixing K pointwise) with image a proper subset of itself. This has to do with the fact that some 1-types over K are realized infinitely often in $\hat{K}$, so that the image of the embedding can omit some of those realizations. As a prime model, the differential closure realizes exactly those 1-types which are principal over K, i.e. generated by a single formula with parameters from K. It therefore omits the type of a differential transcendental over K, since this type is not principal, and so every element of $\hat{K}$ satisfies some differential polynomial over K. On the other hand, the type of a *transcendental constant*, i.e. an element x with $\delta x = 0$ but not algebraic over K, is also non-principal and hence is also omitted, even though such an element would be "differentially algebraic" over K.

The goal of the current work in computable differential fields, by the speaker and two co-authors, is to adapt the two fundamental theorems from computable field theory to computable differential fields. These two theorems, each used very frequently in work on computable fields, are the following.

Theorem 1. *(Kronecker's Theorem (1882); see* [5] *or* [2]*)*

(1) The field $\mathbb{Q}$ has a splitting algorithm. That is, the set of irreducible polynomials in $\mathbb{Q}[X]$, commonly known as the splitting set *of $\mathbb{Q}$, is decidable.*
(2) If a computable field F has a splitting algorithm, so does the field $F(x)$, for every element x algebraic over F (within a larger computable field).
(3) If a computable field F has a splitting algorithm, then so does the field $F(t)$, for every element t transcendental over F.

(The algorithms in Parts II and III are different, and no unifying algorithm exists.)

Theorem 2. *(Rabin's Theorem (1960); see* [7]*)*

(1) Every computable field F has a Rabin embedding*, i.e. a computable field embedding $g : F \to E$ such that E is a computable, algebraically closed field which is algebraic over the image $g(F)$.*
(2) For every Rabin embedding g of F, the image $g(F)$ is Turing-equivalent to the splitting set S_F of F.

For differential fields, the analogue of the first part of Rabin's Theorem was proven in 1974 by Harrington, who showed that for every computable differential field K, there is a computable embedding g of K into a computable, differentially closed field L such that L is a differential closure of the image $g(K)$. Harrington's proof used a different method from that of Rabin, and therefore did not address the question of the Turing degree of the image. Indeed, the first question to address, in attempting to adapt either of these theorems for differential fields, is the choice of an appropriate analogue for the splitting set S_F in the differential context.

Kronecker saw the question of reducibility of a polynomial in $F[X]$ simply as a natural question to ask. With twentieth century model theory, we understand better the reasons why it is important. Specifically, every irreducible polynomial $p(X) \in F[X]$ generates a principal type over the theory $\mathbf{ACF}_0 \cup \Delta(F)$, and every principal type is generated by a unique monic irreducible polynomial. (More exactly, the formula $p(X) = 0$ generates such a type.) On the other hand, no reducible polynomial generates such a type (with the exception of powers $p(X)^n$ of irreducible polynomials, in which case $p(X)$ generates the same type). So the splitting set S_F gives us a list of generators of principal types, and every element of $\overline{F}$ satisfies exactly one polynomial on the list. Moreover, since these generating formulas are quantifier-free, we can readily decide whether a given element satisfies a given formula from the list or not. Thus, a decidable splitting set allows us to identify elements of $\overline{F}$ very precisely, up to their orbit over F.

From model theory, we find that the set $\overline{T_K}$ of *constrained pairs* over a differential field K plays the same role for the differential closure. A pair (p, q) of differential polynomials from $K\{Y\}$ is *constrained* if p is monic and irreducible and of greater order than q (i.e. for some r, $p(Y)$ involves $\delta^r Y$ nontrivially while $q(Y) \in K[Y, \delta Y, \ldots, \delta^{r-1} Y]$) and, for every $x, y \in \hat{K}$, if $p(x) = p(y) = 0$ and $q(x) \neq 0 \neq q(y)$, then there exists $h \in K\{Y\}$ such that either $h(x) = 0 \neq h(y)$ or $h(y) = 0 \neq h(x)$. This says that, if x and y both *satisfy* the pair (p, q), then the differential fields $K\langle x\rangle$ and $K\langle y\rangle$ which they generate within $\hat{K}$ must be isomorphic, via an isomorphism fixing K pointwise and mapping x to y. This is sufficient to ensure that the formula $p(Y) = 0 \neq q(Y)$ generates a principal type over $\mathbf{DCF}_0 \cup \Delta K$, and conversely, every principal type is generated by such a formula with (p, q) a constrained pair. With this background, we may state our results, first addressing Rabin's Theorem and then Kronecker's.

Theorem 3. *For every embedding g of a computable differential field as described by Harrington in* [3]*, the image $g(K)$ is Turing-computable from the set $\overline{T_K}$. So too is algebraic independence of finite tuples from $\hat{K}$, and also the function mapping each $x \in \hat{K}$ to its minimal differential polynomial over K. However, there do exist such embeddings g for which $\overline{T_K}$ has no Turing-reduction to $g(K)$.*

Theorem 4. *Let K be a computable nonconstant differential field, with $z \in \hat{K}$. Then $\overline{T_{K\langle z\rangle}}$ is Turing-computable from $\overline{T_K}$.*

So the middle part of Kronecker's Theorem holds. We believe that we also have a proof for constant fields, and for the third part, but this remains to be checked.

Conjecture 5. Let K be a computable differential field, and z an element differentially transcendental over K within some larger comptuable differential field. Then $\overline{T_{K\langle z\rangle}}$ is Turing-computable from $\overline{T_K}$.

It remains to determine whether the set $\overline{T_{\mathbb{Q}}}$ of constrained pairs over the constant differential field $\mathbb{Q}$ is decidable; we regard this as the most important question currently open in this area of study. A positive answer would likely give us a much better intuition about the structure of various simple differentially closed fields, well beyond any current understanding. It would also be desirable to make the failure of the second part of Rabin's Theorem more precise, by finding sets which are always equivalent to the Rabin image $g(F)$, and by finding sets which are always equivalent to $\overline{T_K}$.

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Recent results on random sets

Frank Stephan

(joint work with Johanna Franklin, Keng Meng Ng, André Nies)

The talk presented recent work done with Johanna Franklin as well as with Keng Meng Ng and André Nies. The common topic was intersecting random sets with r.e. or recursive sets and studying the Turing degree of the resulting sets. Here Martin-Löf random sets and Schnorr random sets were studied; the definitions of these sets are as follows. A martingale is a function M from binary strings into positive real numbers such that $M(\sigma) = 0.5(M(\sigma 0) + M(\sigma 1))$. A set A is Martin-Löf random iff there is no martingale M such that $M(A(0)A(1)\ldots A(n))$ is unbounded and $\{(\sigma, q) : \sigma \in \{0,1\}^*, q \in \mathbb{Q}, M(\sigma) > q\}$ is recursively enumerable; A is Schnorr random iff there is no martingale M and no recursive function f such that $M(A(0)A(1)\ldots A(f(n))) > n$ for infinitely many n and $\{(\sigma, q) : \sigma \in \{0,1\}^*, q \in \mathbb{Q}, M(\sigma) > q\}$ is recursive. A set B is co-retraceable iff there is a recursive function f such that the complement of B, given in ascending order as $\{b_0, b_1, \ldots\}$, satisfies that $f(b_{n+1}) = f(b_n)$ for all n; here it is assumed that $b_0 < b_1 < \ldots$ and one can have that f is total as one can define f arbitrarily when the input is enumerated into B. Co-retraceable r.e. sets are well-studied. They are known to be semirecursive. Dekker's deficiency sets are the most prominent examples of r.e. co-retraceable sets. See standard text books on algorithmic randomness or recursion theory for further information [3, 4, 6]; a draft of the work with Johanna Franklin is available from the speaker's homepage [2] and a draft of the work with Ng and Nies can be found in the logic blog 2012 on Nies' homepage [5].

In joint work with Johanna Franklin, it is shown that if A is a Martin-Löf random and B is r.e., Turing incomplete and coretraceable then $B \leq_T A \cap B$; furthermore, if A is Schnorr random and B is an r.e., non-high and coretraceable set then $B \leq_T A \cap B$. Both results are sharp; that is, there are the following counterexamples: One can choose B Turing complete and co-retraceable and A

Martin-Löf random such that $B \not\leq_T A \cap B$; one can choose B as a r.e. coretraceable set of given high r.e. Turing degree and A as a Schnorr random subset of B strictly Turing below B such that $B \not\leq_T A$ and $A = A \cap B$. Furthemore, one can choose B to be low and r.e. and A to be Martin-Löf random with $B \not\leq_T A \cap B$ in order to see that the condition of B being co-retraceable can also not be omitted from the above theorems. Furthermore, it is shown that the complement of an r.e. coinfinite set B is indifferent for Schnorr randomness iff B is dense simple. More precisely, for every Schnorr random set A and every dense simple set B and every set C with $A \cap B = C \cap B$ it holds that C is Schnorr random; in addition one has that if A is Schnorr random and B is r.e., coinfinite and not dense simple then there is a set C which is not Schnorr random and satisfies $A \cap B = C \cap B$. This result extends work done by Figueira, Miller and Nies on indifference for random sets [1].

In joint work with Keng Meng Ng and André Nies, it is shown that in every high Turing degree there is a Schnorr random set A such that $A \equiv_T A \cap B$ for every infinite recursive set B. Furthermore, for every set C there is a Martin-Löf random set A such that, for every recursive set B, either $A \cap B \geq_T C$ or $A \cap (\mathbb{N} - B) \geq_T C$. This shows that there are Martin-Löf random sets A which cannot be split into halves along any recursive set B such that both halves are Turing incomplete or low or otherwise computationally weak. This insight strengthens the findings of Gács and Kučera who proved that there is a Martin-Löf random set Turing above any given set C. Furthermore, this result does not hold on the level of Turing degrees: If C is Turing complete then there is a low set A and a further Turing incomplete set B such that $A \oplus B \equiv_T C$ and $A \oplus B$ is Martin-Löf random. To prove this, one starts with choosing A to be Martin-Löf random and low; then one chooses relative to A a random set with its jump being B, call it B_0. Furthermore, let $B_1 = \Omega^{A \oplus B_0}$ be Chaitin's Ω relativised to $A \oplus B_0$. One can see by using van Lambalgen's theorem twice that $A \oplus B_0 \oplus B_1$ is Martin-Löf random; furthermore, it has the Turing degree of $(A \oplus B_0)'$ what is the one of C. Now taking $B = B_0 \oplus B_1$ completes the proof, as B is Martin-Löf random relative to A and, as a Martin-Löf random set cannot be a base of Martin-Löf randomness, $A \not\leq_T B$; hence B is Turing incomplete.

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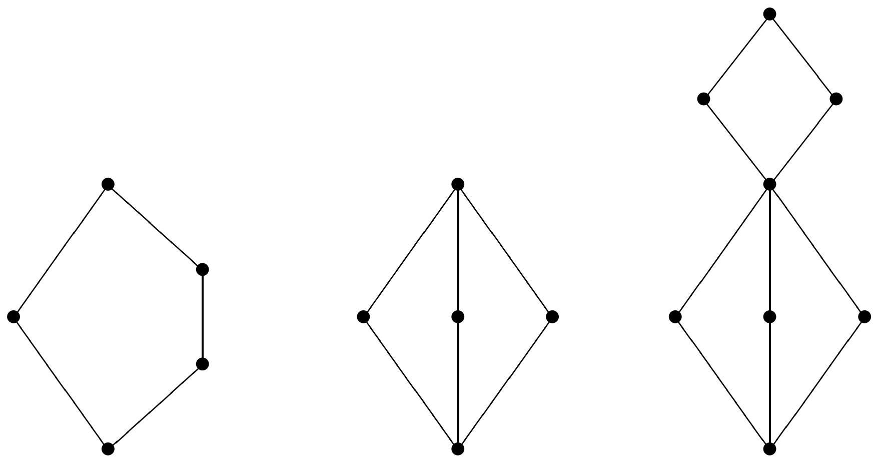
Lattice embeddings into the computably enumerable ibT- and cl-degrees

Thorsten Kräling
(joint work with Klaus Ambos-Spies, Philipp Bodewig, Liang Yu, and Wei Wang)

Given any degree structure $\mathcal{D}$, an interesting question is which (finite) lattices can be embedded into $\mathcal{D}$ preserving joins and meets.

For $\mathcal{R}_{\mathrm{T}}$, the structure of the computably enumerable Turing degrees, this is a long-standing open problem. It is known that every distributive finite lattice can be embedded into $\mathcal{R}_{\mathrm{T}}$ (Lachlan-Lerman-Thomason, see [6]) and that some nondistributive lattices can be embedded. In particular, Lachlan [3] showed that the nondistributive modular 5-element lattice M_3 and the nondistributive non-modular 5-element lattice N_5 (see the diagrams below), are embeddable into $\mathcal{R}_{\mathrm{T}}$. These two lattices are the most basic nondistributive lattices in the sense that every nondistributive lattice contains at least one of them as a sublattice. There are also examples of lattices known which cannot be embedded into $\mathcal{R}_{\mathrm{T}}$, like the 8-element lattice S_8 (see Lachlan and Soare [5]) or a 20-element lattice by Lempp and Lerman [4], but a simple characterisation of which lattices are embeddable has not been found so far.



N_5 M_3 S_8

For other degree structures the picture looks different. For example, Fejer and Shore [2] have shown that every finite lattice can be embedded into $\mathcal{R}_{\mathrm{tt}}$, the structure of the computably enumerable truth-table degrees.

Here, we are looking at the embeddability question with respect to the degree structure $\mathcal{R}_{\mathrm{cl}}$ of the computably enumerable computable Lipschitz degrees and the degree structure $\mathcal{R}_{\mathrm{ibT}}$ of the computably enumerable identity-bounded Turing degrees. A set A is *computably Lipschitz-(cl-)reducible* to a set B if it is Turing-reducible to B via a reduction where the oracle questions to determine $A(x)$ are bounded by $x + c$ for some constant c. If this constant can be chosen to be 0, then A is called *identity-bounded Turing-(ibT-)reducible* to B.

Ambos-Spies [1] observed that every finite distributive lattice can be embedded into $\mathcal{R}_{\text{cl}}$ and $\mathcal{R}_{\text{ibT}}$ preserving the least element. Using more elaborate methods, we were able to establish the following two results about nondistributive lattices.

Theorem 1 (Ambos-Spies, Bodewig, Kräling, and Yu, unpublished)**.** *The N_5 can be embedded into $\mathcal{R}_{\text{cl}}$ and $\mathcal{R}_{\text{ibT}}$ preserving the least element.*

Theorem 2 (Ambos-Spies, Bodewig, Kräling, and Wang, unpublished)**.** *The M_3 can be embedded into $\mathcal{R}_{\text{cl}}$ and $\mathcal{R}_{\text{ibT}}$.*

On the other hand, by a yet unpublished result of Ambos-Spies and Wang the M_3 cannot be embedded into $\mathcal{R}_{\text{cl}}$ or $\mathcal{R}_{\text{ibT}}$ preserving the least element. This shows that embeddability and embeddability preserving $\mathbf{0}$ are not equivalent for $\mathcal{R}_{\text{cl}}$ and $\mathcal{R}_{\text{ibT}}$, while for $\mathcal{R}_{\text{T}}$ the question whether such an equivalence holds is still open. Moreover, since it can be shown that each c.e. ibT- or cl-degree is the bottom of a diamond lattice, Theorem 2 implies the following, once again contrasting the situation in the c.e. Turing degrees.

Corollary 3. *The S_8 can be embedded into $\mathcal{R}_{\text{cl}}$ and $\mathcal{R}_{\text{ibT}}$.*

The general embeddability question for $\mathcal{R}_{\text{cl}}$ and $\mathcal{R}_{\text{ibT}}$ is still open.

Question. *Is there a finite lattice which cannot be embedded into $\mathcal{R}_{\text{cl}}$ or $\mathcal{R}_{\text{ibT}}$?*

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Computably enumerable equivalence relations

Steffen Lempp

(joint work with Uri Andrews, Joseph S. Miller, Keng Meng Ng, Luca San Mauro and Andrea Sorbi)

We investigate computably enumerable equivalence relations, which naturally arise in the context of computable numberings and the investigation of provability from theories of first-order arithmetic.

In particular, we answer two questions of Gao and Gerdes [1] by proving that if the halting jump operator of a c.e. equivalence relation R is of the same degree as R itself, then R is universal, and that universality of c.e. equivalence relations is a Σ^0_3-complete property.

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A Hierarchy of Computably Enumerable Degrees, Unifying Classes and Natural Definability

Rod Downey
(joint work with Noam Greenberg)

1. Introduction

In this lecture, we will discuss recent programme of the authors which is still in a state of formation, but already has several published results (Barmpalias, Downey, Greenberg [2], Day [4], Downey, Greenberg and Weber [9] and Downey and Greenberg [7]) as well as more in preparation (or being worked out!), such as Downey and Greenberg [8], devoted to understanding some new naturally definable degree classes which capture the dynamics of various natural constructions arising from disparate areas of classical computability theory.

It is quite rare in computability theory to find a single class of degrees which capture precisely the underlying dynamics of a wide class of apparently similar constructions, demonstrating that they all give the same class of degrees. A good example of this phenomenon is work pioneered by Martin [17] who identified the high c.e. degrees as the ones arising from dense simple, maximal, hh-simple and other similar kinds of c.e. sets constructions. Another example would be the example of the promptly simple degrees by Ambos-Spies, Jockusch, Shore and Soare [1]. Another more recent example of current great interest is the class of K-trivial reals of Downey, Hirscheldt, Nies and Stephan [6], and Nies [18].

We remark that in each case the clarification of the relevant degree class has lead to significant advances in our basic understanding of the c.e. degrees. We believe the results we mention in the present paper fall into this category. Our results were inspired by another such example, the array computable degrees introduced by Downey, Jockusch and Stob [10, 11]. This class was introduced by those authors to explain a number of natural "multiple permitting" arguments in computability theory. The reader should recall that a degree $\mathbf{a}$ is called array noncomputable iff for all functions $f \leq_{wtt} \emptyset'$ there is a a function g computable from $\mathbf{a}$ such that $\exists^\infty x \ (g(x) > f(x))$.

2. Totally ω-c.a. degrees

Our two new main classes are what we call the *totally ω-c.e. degrees* and the *totally $< \omega^\omega$-c.e. degrees*. These classes turn out to be completely natural and relate to natural definability in the c.e. degrees as we will discuss below. We begin with the ω case.

Definition 1 (Downey, Greenberg, Weber [9]). We say that a c.e. degree $\mathbf{a}$ is *totally ω-c.a.* if for all functions $g \leq_T \mathbf{a}$, g is ω-c.e.. That is, there is a computable approximation $g(x) = \lim_s g(x,s)$, and a computable function h, such that for all x,

$$|\{s : g(x,s) \neq g(x,s+1)\}| < h(x).$$

The reader should keep in mind that array computability is a uniform version of this notion where h can be chosen independent of g. This class captures a number of natural constructions in computability theory.

For example, we can define a class of reals to be finitely boundedly random iff it passes all Martin-Löf tests $\{U_n \mid n \in \omega\}$ where U_n is a clopen set (given by a c.e. index and has at most $g(n)$ many members for some order g.

Theorem 2 (Brodhead, Downey and Ng [3]). *A c.e. degree $\mathbf{a}$ contains a f.b. left c.e. real iff $\mathbf{a}$ is not totally ω-c.a.*

One of the really fascinating things is that this is all connected to *natural* definability issues within the computably enumerable Turing degrees. At the present time, as articulated in Shore [20], there are very few such natural definability results.

In [7, 9], we gave some new natural definability results for the c.e. degrees. Moreover, these definability results are related to the central topic of lattice embeddings into the c.e. degrees as analyzed by, for instance, Lempp and Lerman [15], Lempp, Lerman and Solomon [16].

A central notion for lattice embeddings into the c.e. degrees is the notion of a *weak critical triple.* The reader should recall from Downey [5] and Weinstein [21] that three incomparable elements $\mathbf{a}_0, \mathbf{a}_1$ and $\mathbf{b}$ in an upper semilattice form a weak critical triple if $\mathbf{a}_0 \cup \mathbf{b} = \mathbf{a}_1 \cup \mathbf{b}$ and there is no $\mathbf{c} \leq \mathbf{a}_0, \mathbf{a}_1$ with $\mathbf{a}_0 \leq \mathbf{b} \cup \mathbf{c}$. This notion captures the need for "continuous tracing" which is used in an embedding of the lattice M_5 into the c.e. degrees (first embedded by Lachlan [14]).

The necessity of the "continuous tracing" process was demonstrated by Downey [5] and Weinstein [21] who showed that there are initial segments of the c.e. degrees where no lattice with a (weak) critical triple can be embedded. Downey and Shore [13] prove that if $\mathbf{a}$ is non-low$_2$ then $\mathbf{a}$ bounds a copy of M_5.

Theorem 3 (Downey, Greenberg and Weber [9]). *A degree $\mathbf{a}$ is totally ω-c.a. iff it does not bound a weak critical triple in the c.e. degrees. Hence, the notion of being totally ω-c.a. is naturally definable in the c.e. degrees.*

This class also codes some other constructions. For example:

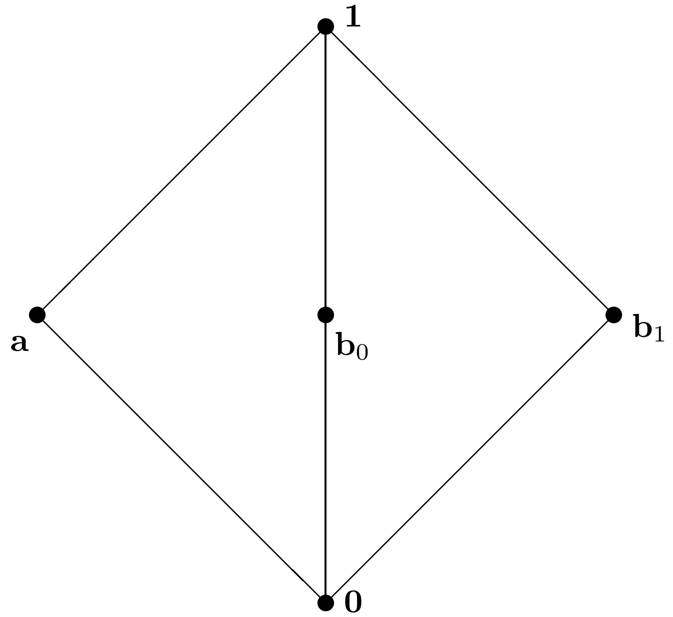

Figure 1. The lattice M_5

Theorem 4 (Downey, Greenberg and Weber [9]). *A c.e. degree* $\mathbf{a}$ *is totally* ω*-c.a. iff there are c.e. sets* A*,* B *and* C *of degree* $\leq_T \mathbf{a}$*, such that*

(i) $A \equiv_T B$
(ii) $A \not\leq_T C$
(iii) *For all* $D \leq_{wtt} A, B$*,* $D \leq_{wtt} C$*.*

3. Totally $< \omega^\omega$-c.e. degrees

The class of totally $< \omega^\omega$-c.a. degrees also arises quite naturally. Recall that if b is an ordinal notation in Kleene's $\mathcal{O}$, then a Δ^0_2 function g is *b-c.a.* if there is a computable approximation $g(x, s)$ for g such that the number of changes in the guessed value is bounded by some decreasing sequence of notations below b; that is, there is a function $o(x, s)$ such that for every x and s, $o(x, s) <_{\mathcal{O}} b$, $o(x, s+1) \leq_{\mathcal{O}} o(x, s)$ and if $g(x, s+1) \neq g(x, s)$ then $o(x, s+1) <_{\mathcal{O}} o(x, s)$. The definition of the class of totally $< \omega^\omega$-c.e. degrees involves *strong notations*, being notations for ordinals in Kleene's sense, except that we ask that below the given notation, Cantor normal form can be effectively computed. Exact formalization of this notion is straightforward for the ordinals below ϵ_0; such notations are computably unique, and so the corresponding class of functions is invariant under the chosen strong notation for a given ordinal; we thus call a function α-c.a. if it is b-c.a. for some (all) strong notations b for α. A degree $\mathbf{a}$ is totally $< \omega^\omega$-c.a. if every $g \leq_T \mathbf{a}$ is ω^n-c.a. for some n. In [7], Downey and Greenberg introduced this notion and in [8] will show that the collection of totally $< \omega^\omega$-c.a. degrees is naturally definable:

Theorem 5 (Downey and Greenberg [8]). *A c.e. degree is totally* $< \omega^\omega$*-c.a. iff it does not bound a copy of* M_5*.*

Again, Downey and Greenberg showed that a number of other constructions gave rise to the same class. Adam Day [4] has shown that this class relates to generic sets which can compute what are called indifferent subsets of themselves. (Namely flipping the bits any way on those positions keeps the real generic.)

We also examine the hierarchy and examine how we can add promptness to get infima **a**0.

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