Unique-maximum and conflict-free coloring for hypergraphs and trees

Panagiotis Cheilaris

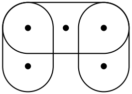
Faculty of Informatics

Università della Svizzera italiana (USI Lugano)

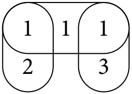
joint work with **Balázs Keszegh** and **Dömötör Pálvölgyi**

Hypergraphs and colorings

Def. A *hypergraph* is a pair $(V, \mathcal{E})$, where $\mathcal{E}$ is a family of subsets of V. An element $e \in \mathcal{E}$ is called a *hyperedge*.

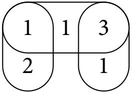


Def. A (vertex) coloring with k colors is a function $C\colon V \to \{1, \ldots, k\}$.



Unique-maximum and conflict-free coloring

Def. A *unique-maximum* (UM) coloring of $H = (V, \mathcal{E})$ is a coloring of H such that for every hyperedge $e \in \mathcal{E}$, the maximum color occurring in e occurrs exactly once in e.



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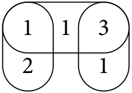
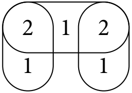


Def. A *conflict-free* (CF) coloring of $H = (V, \mathcal{E})$ is a coloring of H such that for every hyperedge $e \in \mathcal{E}$, there is a color in e which occurs exactly once in e.



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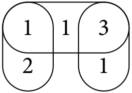

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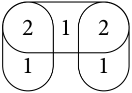

Fact. Every UM coloring is a CF coloring.

Motivation for conflict-free and unique-maximum coloring

(Even, Lotker, Ron, Smorodinsky, 2003)

- Cellular networks consist of fixed position *base stations* (or antennas) that emit at a specific frequency, and *moving agents*.

- Each moving agent has a range of communication that can be modeled by a shape (like a disk). The range includes a subset S of the base stations. We want each such S to contain a base station with ***unique*** frequency in S.

- Model: base stations $\to$ points, frequencies $\to$ colors

- The frequency spectrum is expensive. Therefore, we try to minimize frequency use, i.e., reuse frequencies as much as possible.

Another motivation: Hypergraphs induced by simple paths in graphs

Def. Given $G = (V, E)$, define:

$$H^{\mathrm{p}}(G) = (V, \{S \mid S \text{ is the vertex set of a simple path in } G\}) .$$

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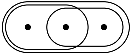
$H^{\mathrm{p}}(P_3)$

Motivation for colorings of $H^{\mathrm{p}}(G)$

Unique-maximum coloring of $H^{\mathrm{p}}(G)$ is also known as *ordered coloring* or *vertex ranking* of graph G.

Applications of efficient (few colors) vertex rankings:

- parallel Cholesky decomposition of matrices (Liu, 1990)

- planning efficient assembly of products in manufacturing systems (Iyer, Ratliff, Vijayan, 1988)

- In general, the unique-maximum coloring problem can model situations where interrelated tasks have to be accomplished fast in parallel (assembly from parts, parallel query optimization in databases, etc.)

- worst-case complexity of finding local optima in neighborhood structures (Llewellyn, Tovey, Trick, 1989)

Chromatic numbers

Def. The minimum k such that H has a UM-coloring with k colors is called the UM-*chromatic number* of H, denoted by $\chi_{\mathrm{um}}(H)$.

Def. The minimum k such that H has a CF-coloring with k colors is called the CF-*chromatic number* of H, denoted by $\chi_{\mathrm{cf}}(H)$.

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Shorthand notation:
$UM(G) = \chi_{\mathrm{um}}(H^{\mathrm{p}}(G))$ and $CF(G) = \chi_{\mathrm{cf}}(H^{\mathrm{p}}(G))$

Our results in this work

How bigger than $\chi_{\mathrm{cf}}(H)$ can $\chi_{\mathrm{um}}(H)$ be for the same hypergraph H?

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Thm. For $H = (V, \mathcal{E})$ with $n = |V|$,

$$\chi_{\mathrm{um}}(H) \leq \lceil n/\chi_{\mathrm{cf}}(H) \rceil + 1$$

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For hypergraphs $H^p(T)$ induced by paths of some tree graph T:

Thm. For any tree T, $[UM(T) = \chi_{um}(H^p(T)), CF(T) = \chi_{cf}(H^p(T))]$

$$UM(T) \leq (CF(T))^3 + o((CF(T))^3)$$

and there are trees for which $UM(T) \simeq 1.58 \cdot CF(T)$.

Our results in perspective

	upper bound	biggest um/cf difference
hypergraph		
graph		
tree		

Reminder:

$$UM(G) = \chi_{\mathrm{um}}(H^{\mathrm{p}}(G)), \quad CF(G) = \chi_{\mathrm{cf}}(H^{\mathrm{p}}(G))$$

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graph	$\forall G\colon UM(G) \leq 2^{CF(G)} - 1$	$\exists G\colon UM(G) \simeq 2 \cdot CF(G)$
tree	$\forall T\colon UM(T) \leq (CF(T))^3$	$\exists T\colon UM(T) \simeq 1.58 \cdot CF(T)$

In [Ch., Tóth, 2011], we proved:

Thm. For every graph G,

$$UM(G) \leq 2^{CF(G)} - 1$$

and there are graphs for which $UM(G) \simeq 2 \cdot CF(G)$.

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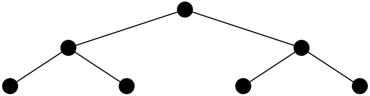
$$UM(G) = \chi_{um}(H^p(G)), \quad CF(G) = \chi_{cf}(H^p(G))$$

Notions used in tree upper bound proof

Def. The *path graph* on n vertices (denoted by P_n).

Def. The *complete binary tree* with d levels (denoted by B_d). It has $2^d - 1$ vertices.

Example: B_3



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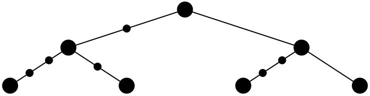

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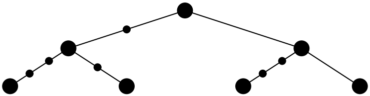

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Def. Odd coloring and $ODD(G)$.

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Lemma. $CF(P_n) = \lceil \log_2 (n+1) \rceil$

Lemma. Let B_d be the complete binary tree with d levels. Then, for any subdivision B^* of B_d, we have $CF(B^*) \geq \sqrt{d}$.

Trees for which CF is smaller than UM

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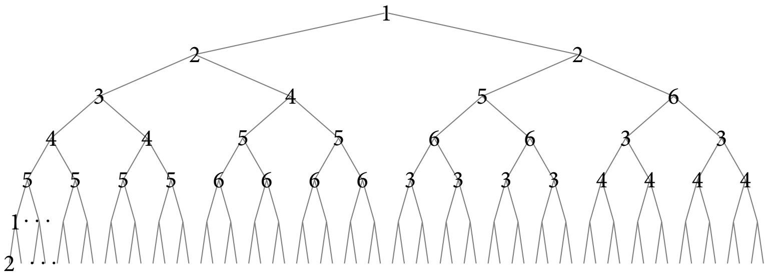

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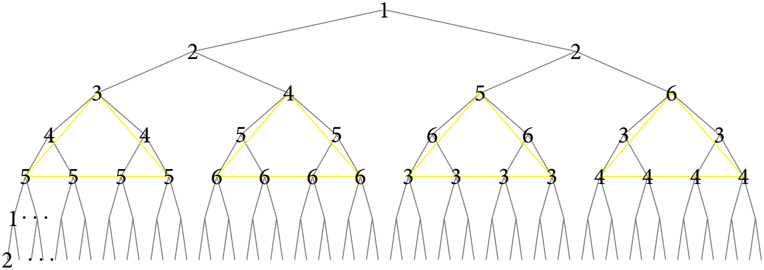

The colors in the four B_3 subtrees at levels 3 to 5:

$$\{(3,4,5),(4,5,6),(5,6,3),(6,3,4)\}$$

Prefix-Set-Free (PSF) families and CF colorings

Families like $\{(3,4,5),(4,5,6),(5,6,3),(6,3,4)\}$ are called *Prefix-Set-Free* (PSF).

We introduce and study them in this work.

Def. A family $\mathcal{F}$ of ordered sets is said to be *PSF* if any prefix of any ordered set in $\mathcal{F}$ is different from any other set in $\mathcal{F}$ without its order.

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We use PSF families to find efficient CF colorings of complete binary trees.

Thm. For the sequence of complete binary trees $\{B_i\}_{i=1}^{\infty}$, the limit of the ratio of the UM to the CF number is at least $log_2 3 \simeq 1.58$.

Related work and open problems

UM and CF colorings of graphs w.r.t. paths (joint work with Géza Tóth)

close the gap between UM and CF for the classes of graphs and tres

monotonicity of CF under the subdivision operation?

list UM and CF colorings of hypergraphs (joint work with Marek Sulovský and Shakhar Smorodinsky)

Thank you!