Ordinal Logics and Proof Theory

G. Mints
Stanford University
AMS-ASL Special Session
on the Life and Legacy of Alan Turing
January 5, 2011

Ordinal Logics

Turing (1939)

"The well-known theorem of Gödel's (1931) shows that every system of logic is in a certain sense incomplete, but at the same time it indicates means whereby from a system L of logic a more complete system L' may be obtained. By repeating the process we get a sequence

$$L, L_1 = L', L_2 = L'_1, \ldots$$

each more complete than the preceding. A logic L_ω may then be constructed in which the provable theorems are the totality of theorems provable with the help of the logics $L, L_1, L_2, \ldots$ Proceeding in this way we can associate a system of logic with any constructive ordinal. It may be asked whether such a sequence of logics of this kind is complete in the sense that to any problem A there corresponds an ordinal α such that A is solvable by means of the logic L_α."

In fact L' is obtained by adding to L a consistency statement Con_L.

Turing gave a positive answer for purely universal (Π^0_1) sentences.

Theorem 1 (Turing) *For each true Π^0_1 sentence ϕ we can find a constructive ordinal a with $|a| = \omega + 1$ and $L_\alpha \vdash \phi$.*

Constructive ordinals:

$|1| = 0$;

$|2^a| = |a| + 1$;

$|3 \cdot 5^e| = \lim_n |a_n|$,

provided $|\{e\}(n)|$ is increasing.

S. Feferman (1962) extended this result to arbitrary arithmetical sentences, but reflection principles instead of consistency statements are needed there.

Autonomous Progressions

Two foundational conceptions of a mathematical proof:

finitist (Hilbert) and predicative (Poincare).

If a logical system L is acceptable under some foundational conception, then it is of course consistent. Therefore

$L' = L \cup Con_L$ is acceptable too.

Also, if each of a sequence

$$L_{a_1} \subset L_{a_2} \subset \ldots \subset L_{a_n} \subset \ldots$$

of increasing systems is recognized to be acceptable then

$$\cup_n L_{a_n}$$

should be acceptable too.

The only problem: for which representations a of ordinals it is justified to accept L_a ?

Kreisel-style Analysis of Finitism

Kreisel's (1959) solution: require

$L_b \vdash$ (a is an ordinal) for some $b < a$.

Finitist standpoint assumes as given and immediately understood combinatorial objects (completely coded by natural numbers) and computable functions of these objets.

One interpretation of this (especially in Hilbert-Bernays) allows existential (Σ^0_1) formulas and free variables for natural numbers.

G. Kreisel (1964) allows also free variables for numerical functions and provides in this Σ^0_1 framework a construction of a system of the strength of Peano arithmetic PA to justify thesis:

Finitism = PA.

As far as I know, complete proof of equivalence to PA for the system introduced by Kreisel was never published.

Let's describe a similar construction of the $\exists$-part of PA.

Now ordinals are represented by primitive recursive linear orderings $\prec$.

The condition

"$\prec$ is well-founded" $=$ no infinite descending sequences

$$f(0) \succ f(1) \succ \ldots \succ f(n) \succ f(n') \succ \ldots$$

is expresed by an existential formula

$$\exists x \neg f(x') \prec f(x)$$

with free function variable f.

System KPA

Formulas of KPA: arithmetical Σ_1^0 formulas with free function variables.

$(\&, \vee)$-combinations and bounded quantifiers: abbreviations.

Function terms: constants for initial primitive recursive functions, free function variables and the results of primitive recursive definitions.

The axioms of KPA are ordinary postulates of first order logic restricted to this language, Peano axioms.

Inference rules: suitable restriction of induction and

$$\frac{WF \;\; Desc \;\; A0 \;\; (\&_{i \leq n} A(\bar{t}_i, u_i, z_i)) \;\; \rightarrow A(\bar{x}, y, Z)}{\exists z A(\bar{x}, y, z)} \, \mathsf{TI}$$

where

$$WF :\equiv \; \exists x f x \not\prec f x'$$

$$Desc :\equiv \; \&_{i \leq n} u_i \prec y$$

$$A0 :\equiv \; \exists z A(\bar{x}, 0, z)$$

and $\prec$ is a primitive recursive order relation, f a function variable, $\bar{t}_i, u_i, Z$ terms, $z_i, \bar{x}$ variables and for any $i \leq n$ the variable z_i does not occur in $\bar{t}_j, u_j$ for $j \leq i$.

PA is a conservative extension of KPA

TI follows from more standard transfinite induction for ∀∃ formulas:

$$\frac{(\forall u \prec y)\forall\bar{x}\exists z A(\bar{x},u,z) \to \forall\bar{x}\exists z A(\bar{x},y,z)}{\forall y\forall\bar{x}\exists z A(\bar{x},u,z)} \qquad (1)$$

TI for each provable well-ordering of PA is provable in PA.

$PA \cap \Sigma^0_1$ is contained in KPA

The rule TI is enough to justify the schema of *nested recursion* (W. Tait 1959) on a relation $\prec$:

$$\begin{aligned}\phi(\bar{x}, 0) &= \psi(\bar{x}) \\ \phi(\bar{x}, y) &= \pi(\bar{x}, y, \lambda z \prec y \lambda \bar{x} \phi(\bar{x}, z))\end{aligned}$$

for $y \neq 0$, where ψ, π are given functions.

$<_n$: $\omega^{\cdot^{\cdot^{\omega}}}$ n times.

Provable recursive functions of PA are defined by recursions on $<_n$ for n=1,2,. . . .

By Tait (1959) functions defined by nested recursion on $<_n$ are sufficient to prove well-foundedness of $<_{n+1}$.

$$\mathsf{KPA} \vdash \exists z F(\bar{x}, z)$$

for graphs of all provable recursive functions of PA, and hence all Σ^0_1-formulas provable in PA, as required.