Definable relations in the Turing degree structures

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Ershov Hierarchy

The finite level $n, n \geq 1$, of the Ershov hierarchy constitutes n-c. e. sets which can be presented in the canonical form as

$$A = \bigcup_{i=0}^{\left[\frac{n-1}{2}\right]} \{(R_{2i+1} - R_{2i}) \cup (R_{2i} - R_{2i+1})\}$$

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$R_0 \subseteq R_1 \subseteq R_2 \subseteq ... \subseteq R_{n-1}$.
(Here if n is an odd number then $R_n = \emptyset$.)
$R_0,\ R_1 - R_0,\ (R_2 - R_1) \cup R_0, \ldots.$

A Turing degree **a** is n-c. e. if it contains an n-c. e. set, and it is a *properly n-c. e. degree* if it contains an n-c. e. set but no $(n-1)$-c. e. sets.

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We denote by $\mathcal{D}_n$ the partial ordered set of all n-c. e. degrees and by $\mathcal{D}$ the class of all Turing degrees. $\mathcal{R} = \mathcal{D}_1$ denotes the set of c. e. degrees.

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$\mathcal{R} = \mathcal{D}_1$ denotes the set of c. e. degrees.

$$\mathcal{R} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \ldots \subset \mathcal{D}_n \subset \ldots \subset \mathcal{D}(\leq \mathbf{0}')$$

Open questions

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more specifically, questions related to the definability of the relations of ‘computably enumerable’ and ‘computably enumerable in’;

Definability in the degree structures

We say that a set of Turing degrees $\mathcal{C}$ is definable in a structure $\mathcal{D}_n$, if there is a formula $\varphi(x)$ in its language $\mathcal{L} = \{\leq\}$ such that

$$\mathcal{D}_n \models \varphi(\mathbf{a}) \text{ iff } a \in \mathcal{C}$$

.

Known results. The case of c.e. degrees.

Let for each $n > 0$,

$$H_n = \{\mathbf{a} \text{ c.e.} \mid \mathbf{a^n} = \mathbf{0^{(n+1)}}\},$$
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In c.e. degrees:

- All classes H_n and L_n (except possibly L_1) are definable. (Nies, Shore and Slaman)

Known result. The case d-c.e. degrees.

The formula
$\varphi(x) = (\exists \mathbf{y} > \mathbf{x})(\forall \mathbf{z} \leq \mathbf{y})(\mathbf{z} \leq \mathbf{x} \vee \mathbf{x} \leq \mathbf{z})$
defines in $\mathcal{D}_2$ an infinite set of c.e. degrees.(A., Kalimullin, Lempp)

Theorem (Arslanov, Yamaleev)

(*a*) For any *properly* d-c.e. degree and for any nontrivial splitting of $\mathbf{d} = \mathbf{d_0} \cup \mathbf{d_1}$ into d-c.e. degrees $\mathbf{d_0}$ and $\mathbf{d_1}$, for an $i \leq 1$ any d-c.e. degree $\mathbf{u}_i$, $\mathbf{d_i} < \mathbf{u_i} \leq \mathbf{d}$, is splittable in d-c.e. degrees avoiding the upper cone of $\mathbf{d_i}$.

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(*b*) There is a high c.e. degree $\mathbf{c} > \mathbf{0}$ and a nontrivial splitting of $\mathbf{c} = \mathbf{c_0} \cup \mathbf{c_1}$ into low d-c.e. degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ such that for each $i \leq 1$ there is a d-c.e. degree $\mathbf{d_i}, \mathbf{c_i} < \mathbf{d_i} < \mathbf{c}$, which is not splittable into d-c.e. degrees avoiding the upper cone of $\mathbf{c_i}$.

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Conjecture. The class $\mathcal{C}$ dense in the low c.e. degrees: $(\forall \text{ low c.e. } \mathbf{a} < \mathbf{b})(\exists \mathbf{c} \in \mathcal{C})(\mathbf{a} < \mathbf{c} < \mathbf{b})$.

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Theorem (Sacks). Every noncomputable c.e. degree $\mathbf{a}$ is nontrivially splittable into c.e. low degrees: $\mathbf{a} = \mathbf{a_0} \cup \mathbf{a_1}, \mathbf{a}'_\mathbf{0} = \mathbf{a}'_\mathbf{1} = \mathbf{0}'$, and $\mathbf{a_0} \mid \mathbf{a_1}$.

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Low density + Low splitting $\rightarrow$ definability of c.e. degrees in $\mathcal{D}_2$.

Low Density Conjecture

For any given c.e. degree $\mathbf{a} > \mathbf{0}$

- First split $\mathbf{a}$ into two low c.e. degrees $\mathbf{a_0}$ and $\mathbf{b_0}$:

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Then find low c.e. degrees $\mathbf{a_1}$ and $\mathbf{b_1}$ such that
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Then by the Low Density Conjecture find $\mathcal{C}$-degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ such that $\mathbf{a_0} < \mathbf{c_0} < \mathbf{a_1}$ and $\mathbf{b_0} < \mathbf{c_1} < \mathbf{b_1}$.

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We have $\mathbf{a} = \mathbf{c}_0 \cup \mathbf{c}_1$.

A partial result

Theorem. Let $\mathbf{a}$ be a high c.e. degree ($\mathbf{a}' = \mathbf{0}''$). Then there exists a c.e. degree $\mathbf{c} \leq \mathbf{a}$ and a nontrivial splitting of $\mathbf{c}$ into low d-c.e. degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ such that

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for each $i \leq 1$ there exists a d-c.e. degree $\mathbf{d_i}, \mathbf{c_i} < \mathbf{d_i} < \mathbf{c}$, which is not splittable into d-c.e. degrees avoiding the upper cone of $\mathbf{c_i}$.

Open question

Can we change in this theorem the condition $\mathbf{a}' = \mathbf{0}''$ ($\mathbf{a}$ is a high degree) by the condition $\mathbf{a}' = \mathbf{0}'$ ($\mathbf{a}$ is a low degree)?

Theorem. Let $\mathbf{a} > \mathbf{0}$ be a low n-c.e. degree for some $n \geq 1$. Then the set of n-c. e. degrees $\{\mathbf{b} \mid \mathbf{b} \leq \mathbf{a}\}$ is definable from parameters in $\mathcal{D}(\leq \mathbf{0}')$.

Proof.

Theorem. For any low n-c.e. degree $\mathbf{a} > \mathbf{0}$ and for any $n, 1 \leq n < \omega$, the set of all n-c. e. degrees $\{\mathbf{b} \mid \mathbf{b} \leq \mathbf{a}\}$ forms a uniformly low set.

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A set of degrees $\mathcal{A}$ is uniformly low if there exists a sequence of sets $\langle X(n) \mid n \in \omega \rangle$ which are representatives for the degrees in $\mathcal{A}$ (i. e. $\{deg(X(n)) \mid n \in \omega\} = \mathcal{A}$) and there is a $\emptyset'$-computable function f such that $\Phi_{f(n)}^{\emptyset'} = (X(n))'$.

Theorem (Slaman, Woodin). Every uniformly low set of Δ^0_2-degrees, bounded by a low degree **a**, is definable from parameters in $\mathcal{D}(\leq \mathbf{0}')$.

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Is the set of all c.e. degrees definable from parameters in the n-c.e. degrees for some /each $n > 1$?

An additional predicate "x is c. e. in y"

Theorem. The classes of all c.e. and all d-c.e. degrees are definable in $\{\mathcal{D}_n, \leq, \text{CEIN}\}$ for each $n \geq 3$.
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(a) $\mathbf{x}$ is c. e. iff $\mathbf{x}$=CEIN($\mathbf{0}$),

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Proof.
(a) $\mathbf{x}$ is c. e. iff $\mathbf{x}$=CEIN($\mathbf{0}$),

(b) $\mathbf{x}$ is 2-c. e. iff
$(\exists \mathbf{y})(\mathbf{y}$ is c. e., $\mathbf{y} \leq \mathbf{x}$ and $\mathbf{x}$=CEIN($\mathbf{y}$)).

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Proof.
(a) **x** is c. e. iff **x**=CEIN(**0**),

(b) **x** is 2-c. e. iff
$(\exists \mathbf{y})(\mathbf{y}$ is c. e., $\mathbf{y} \leq \mathbf{x}$ and $\mathbf{x}=\text{CEIN}(\mathbf{y}))$.

By Arslanov, LaForte and Slaman, if a n-c.e. ($n \geq 3$) degree **a** CEIN(**b**) for a c.e. degree **b**, then **a** is a d-c.e. degree.

Open question. Is this true for the classes of n-c.e. degrees for $n > 2$?

Definability in $\mathcal{E}_n$, $\mathcal{E}_\infty = \{\bigcup_{1\le n<\omega} \mathcal{E}_n; \cup, \cap, \omega, \emptyset\}$

Let $\mathcal{E}_n, 1 \le n < \omega$, be the class of all n-c.e. sets.
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where $A =^* B$ iff $(A - B) \cup (B - A)$ finite.
Let $\{V_{n,e}\}_{e \in \omega}$ denotes an effective enumeration of all n-c.e. sets, $n \ge 2$.

We say that a set of Turing degrees $\mathcal{C}$ is definable in $\mathcal{E}_\infty$ (in $\mathcal{E}_n$ for some $n \geq 1$) if there is a a definable in $\mathcal{E}_\infty$ (in $\mathcal{E}_n$) class of sets $S \subset \mathcal{E}_\infty$ such that

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Example. The degree $\mathbf{0}$ definable in each $\mathcal{E}_\alpha, 1 \leq \alpha \leq \omega$: $\mathbf{0} = \deg(\emptyset)$.

Known results. C.e. case.

Theorem. In $\mathcal{E} = \mathcal{E}_1$

(a) all classes H_m, $m \geq 1$, are definable (Martin ($m = 1$), Cholak-Harrington ($m > 1$)). and

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(b) all classes $\bar{H}_m$, $m \geq 1$, are non definable (Harrington-Soare, Cholak-Downey-Stob).

Theorem. In $\mathcal{E}$

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(b) all classes $\overline{L}_n, n > 1$, are definable (Lachlan,Shoenfield ($n = 2$), Cholak-Harrington ($n \geq 2$)).

(c) $\overline{L}_1$ not definable (R.Epstein).

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Theorem. The class of all computable sets definable in each $\mathcal{E}_\alpha, 1 \leq \alpha \leq \omega$.

Theorem. For each $n > 0$, the class of sets $\delta_n := \mathcal{E}_{n+1} \cap \bar{\mathcal{E}}_{n+1}$ definable in $\mathcal{E}_{n+1}$.

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Theorem (S.Lempp and A.Nies for the case $n = 2$). The class $\mathcal{E}$ of c.e. sets definable in each $\mathcal{E}_n$, $n > 1$.

Corollary. For all pairs $m, n, 1 \leq m < n$, the class of sets $\mathcal{E}_m$ definable in $\mathcal{E}_n$.

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$H_1 = \{\deg(V_{n,e}) : V_{n,e} \text{ cohesive}\}$.

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Theorem. For each $n > 1$ there exists a non-trivial definable in $\mathcal{E}_n$ set of n-c.e. degrees $\mathcal{C}$ such that $\mathcal{C}$ contains all high n-c.e. degrees.