The Halting Problem Revisited

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The Incomputable, Chicheley Hall, 14 June 2012

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- does not reveal "the cause" of the impossibility.

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- we interest ourselves to (Turing) machines working with natural numbers as inputs and outputs;
- there exists a halting machine **HALT** which solves the halting problem for the above class of machines.

An information-theoretical argument

Construct the machine Trouble(N):

1. `read a natural` N;
2. `generate all machines and inputs` (T, n) `of up to` N `bits in size;`
3. `use` **HALT** `to remove all pairs` (T, n) `for which` T `does not stop on` n;
4. `run the remaining computations` $T(n)$ `till they stop;`
5. `compute the largest value` o `output by these machines and output` $2o + 1$.

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3. For large enough N, Trouble(N) has less than N bits in size.
4. For large enough N, Trouble(N) *generates itself* at some stage of the computation: by examining the output, we get a contradiction.

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- h is incomputable;
- the quantity of information in $h(\restriction n) = h(1)h(2)\cdots h(n)$ is about $\log n$, hence infinitely many bits of h can be computed.

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- the quantity of information in $\Omega_U(\restriction n) = \omega_1\omega_2\cdots\omega_n$ is about n;
- Ω_U is bi-immune, i.e. no Turing machine can compute more than finitely many scattered bits of Ω_U.

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For every universal prefix-free Turing machine U and natural $N > 0$, we can effectively construct another universal prefix-free Turing machine W such that:

1. $\Omega_W = \Omega_U$,
2. given W, ZFC can compute at most $N - 1$ bits of Ω_U, where the first bit of Ω_U equal to zero appears on the position N;if $\Omega_U < 1/2$, then ZFC cannot calculate any bit of Ω_U.

A probabilistic solution for the halting problem

Assume that U is a universal prefix-free Turing machine. We can effectively calculate a stopping time $s = s_U$ such that if $U(x)$ halts in time $t > s_U$, then t is not algorithmically random.

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- "T halts on x", and in this case the result is correct, or
- "T does not halt on x", and in this case the result may be wrong, but with probability less than an arbitrarily small number.

References

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