Effective Categoricity of Injection Structures

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The Incomputable

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Collaborators

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Background

- A computable structure $\mathcal{A}$ is said to be *computably categorical* if any computable structure $\mathcal{B}$ which is isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$
- A computable structure $\mathcal{A}$ is said to be Δ_2^0*categorical* if any computable structure $\mathcal{B}$ which is isomorphic to $\mathcal{A}$ is Δ_2^0 isomorphic to $\mathcal{A}$.
- The computable categoricity of many interesting structures has been studied.
- Computable categoricity of abelian groups by Goncharov (Algebra and Logic 1975)
- Computable categoricity of linear orderings (Proc AMS 1981) and of Boolean algebras (JSL 1981) by Remmel

Previous Work on Δ^0_2 Categoricity

- McCoy studied Boolean algebras and linear orderings (APAL 2003)
- Calvert, Cenzer, Harizanov and Morozov studied equivalence structures (APAL 2006) and Abelian groups (APAL 2009)

Countable Injection Structures

- $\mathcal{A} = (A, f)$
  where A is a set and
  $f : A \to A$ is an injection
- The orbit of a is

$$\mathcal{O}_f(a) = \{b \in A : (\exists n \in \mathbb{N})(f^n(a) = b \ \vee \ f^n(b) = a)\}.$$

- The order of a is

$$|a|_f = card(Or_f(a))$$

- The *character* of $\mathcal{A}$ is

$$\chi(\mathcal{A}) = \{(k, n) : \mathcal{A} \text{ has at least } n \text{ orbits of size } k\}.$$

- $Fin(\mathcal{A}) = \{a : [a] \text{ is finite}\}$
  $Inf(\mathcal{A}) = \{a : [a] \text{ is infinite}\}$

Infinite Orbits

- Injection structures (A, f) may have two types of infinite orbits
- $\mathbb{Z}$-orbits are isomorphic to $(\mathbb{Z}, S)$
  in which every element is in the range of f
- ω-orbits are isomorphic to (ω, S) and have the form $\mathcal{O}_f(a) = \{f^n(a) : n \in \mathbb{N}\}$ for some $a \notin Rng(f)$.
- Thus injection structures are characterized (up to isomorphism) by the character as well as the number of orbits of types $\mathbb{Z}$ and ω.

Complexity of the Orbits and Character

- Let $\mathcal{A} = (\omega, f)$ be a computable injection structure
- Each infinite orbit is a Σ_1^0 set
- **Lemma**

  (a) $\{(k, a) : a \in Rng(f^k)\}$ is a Σ_1^0 set,
  (b) $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ is a Σ_1^0 set,
  (c) $\{a : \mathcal{O}_f(a)$ is infinite$\}$ is a Π_1^0 set,
  (d) $\{a : \mathcal{O}_f(a)$ has type $Z\}$ is a Π_2^0 set,
  (e) $\{a : \mathcal{O}_f(a)$ has type $\omega\}$ is a Σ_2^0 set, and
  (f) $\chi(\mathcal{A})$ is a Σ_1^0 set.

Existence

- **Proposition** For any Σ_1^0 character K, there is a computable injection structure $\mathcal{A}$ such that
  1. $\chi(A) = K$
  2. $Fin(\mathcal{A})$ is computable.
  3. $\mathcal{A}$ may have any specified countable number of orbits of types ω and $\mathbb{Z}$.

Categoricity

- Let $\mathcal{A}$ be a computable structure.
  1. $\mathcal{A}$ is *computably categorical* if, any computable structure which is isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$;
  2. $\mathcal{A}$ is Δ^0_α-categorical if any computable $\mathcal{B}$ isomorphic to $\mathcal{A}$ is Δ^0_α isomorphic to $\mathcal{A}$;
  3. $\mathcal{A}$ is relatively *computably categorical* if, for any computable $\mathcal{B}$ which is isomorphic to $\mathcal{A}$, there exists an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ which is computable from $\mathcal{B}$.
  4. $\mathcal{A}$ is relatively Δ^0_α*-categorical* if, any structure $\mathcal{B} \cong \mathcal{A}$, there is an isomorphism which is Δ^0_α-computable from $\mathcal{B}$.
- Relative categoricity implies categoricity but the converse does not hold in general

Computably Categorical Structures

- **Theorem 1** $\mathcal{A}$ is computably categorical if and only $\mathcal{A}$ has only finitely many infinite orbits.
- **Sketch:** The categoricity follows from the following fact: If $\mathcal{A}$ has finitely many infinite orbits then both *Fin*($\mathcal{A}$) and *Inf*($\mathcal{A}$) are Σ_1^0 and hence both are computable.

  The other direction is sketched below

Scott Families

- A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of $L_{\omega_1\omega}$ formulas, possibly with finitely many fixed parameters from A, such that:
  - (i) Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$;
  - (ii) If $\overrightarrow{a}$, $\overrightarrow{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the *same* formula in Φ, then there is an automorphism of $\mathcal{A}$ that maps $\overrightarrow{a}$ to $\overrightarrow{b}$.
- **Theorem** Let $\mathcal{A}$ be a computable structure. Then the following are equivalent:
  - (a) $\mathcal{A}$ is relatively Δ^0_α categorical;
  - (b) $\mathcal{A}$ has a c.e. Scott family consisting of computable Σ_α formulas.

Relative Computable Categoricity

- **Theorem 2** A computable injection structure $\mathcal{A}$ is relatively computably categorical if and only $\mathcal{A}$ has finitely many infinite orbits.
- **Sketch:** For parameters take one element from each of the infinite orbits. The Scott formula for a sequence $(a_1, \dots, a_m)$ of elements states whether a_j in one of those infinite orbits or has finite order and also for any $i \in \omega$, whether $f^i(a) = b$ for a and b taken from $a_1, \dots, a_m$ plus the parameters.

The Other Direction

- If $\mathcal{A}$ infinitely many infinite orbits, then in fact $\mathcal{A}$ is not computably categorical.
  There are two cases.
- First suppose that $\mathcal{A} = (\omega, f)$ has infinitely many orbits of type ω.
  We may assume that $Rng(f)$ is a computable set.
  Then we build $\mathcal{B} = (\omega, g)$ isomorphic to $\mathcal{A}$ such that $Rng(g)$ is not computable.
- Next suppose that $\mathcal{A}$ has infinitely many orbits of type $\mathbb{Z}$.
  We may assume that each orbit of $\mathcal{A}$ is computable.
  Then we build $\mathcal{B}$ isomorphic to $\mathcal{A}$ with a particular non-computable orbit.

A Corollary

- **Corollary** A computable injection structure $\mathcal{A}$ is relatively computably categorical iff $\mathcal{A}$ is computably categorical.

Δ^0_2 categorical structures

- **Theorem 3** Suppose $\mathcal{A}$ either does not have infinitely many orbits of type ω or does not have infinitely many orbits of type $\mathbb{Z}$. Then $\mathcal{A}$ is relatively Δ^0_2 categorical.
- **Sketch:** Since $Fin(\mathcal{A})$ is a c. e. set and each infinite orbit is also a c. e. set, there is a Δ^0_2 partition of $\mathcal{A}$ into three sets: $Fin(\mathcal{A})$, the orbits of type ω (A_ω), and the orbits of type $\mathbb{Z}$ ($A_\mathbb{Z}$); similarly partition $\mathcal{B}$.
- We can construct isomorphisms between the three parts of $\mathcal{A}$ and the corresponding parts of $\mathcal{B}$.
- For the orbits of type ω, note that the set of beginning elements of orbits is simply $\omega \setminus Rng(f)$ and is therefore a Π^0_1 set.

Non-Δ^0_2 Categorical Structures

- **Theorem 4** If $\mathcal{A}$ has infinitely many orbits of type ω and infinitely many orbits of type $\mathbb{Z}$, then $\mathcal{A}$ is not Δ^0_2 categorical.
- **Sketch:** Consider structures with infinitely many orbits of type ω, infinitely many orbits of type $\mathbb{Z}$, and no finite orbits.
  There is a computable structure $\mathcal{A}$ such that $\mathcal{A}_\omega$ and $\mathcal{A}_\mathbb{Z}$ are computable sets.
  Build a computable structure $\mathcal{B}$ such that $\mathcal{B}_\omega$ is *not* Δ^0_2.
  Then $\mathcal{A}$ and $\mathcal{B}$ are not Δ^0_2 isomorphic.

The Construction

- Let C be an arbitrary Σ_2^0 set.
- We will define g such that $\mathcal{O}_g(2i+1)$ has type ω if and only if $i \in C$.
- The orbits of $\mathcal{B} = (B, g)$ will be exactly $\{\mathcal{O}_g(2i+1) : i \in \mathbb{N}\}$.
- There is a computable function ϕ such that $i \in C$ iff $W_{f(i)}$ is finite.
  Every time a new element comes into $W_{f(i)}$ extend the orbit of $2i+1$ to the left.

A Corollary

- **Corollary** A computable injection structure $\mathcal{A}$ is relatively Δ_2^0 categorical iff $\mathcal{A}$ is Δ_2^0 categorical.

Δ_3^0 Categoricity

- **Theorem** Every computable injection structure is relatively Δ_3^0 categorical.

Computably Enumerable Degrees

- The proof of Theorem 1 has the following corollaries.
- **Corollary** Let **d** be a c. e. degree.
  1. If $\mathcal{A}$ is a computable injection structure which has infinitely many orbits of type ω, then there is a computable injection structure $\mathcal{B} = (B, g)$ isomorphic to $\mathcal{A}$ in which $Rng(g)$ is a c. e. set of degree **d**.
  2. If $\mathcal{A}$ is a computable injection structure which has infinitely many infinite orbits of type $\mathbb{Z}$, then there is a computable injection structure $\mathcal{B} = (B, g)$ isomorphic to $\mathcal{A}$ in which $\mathcal{O}_g(1)$ is of type $\mathbb{Z}$ and is a c. e. set of degree **d**.

The Complexity of $Fin(\mathcal{A})$

- $Fin(\mathcal{A})$ is always a c.e. set but emphcannot be an arbitrary c.e. set
- If $\mathcal{A}$ has an infinite orbit, then this orbit will be an infinite c.e. set in the complement of $Fin(\mathcal{A})$.
- Thus $Fin(\mathcal{A})$ cannot be a simple c.e. set.
- There is a c.e. set which is not simple but is an orbit.

The Degree of $Fin(\mathcal{A})$

- **Theorem** Let **c** be a c. e. degree.
  Let $\mathcal{A} = (A, f)$ be a computable injection structure such that $Fin(\mathcal{A})$ is infinite, $\mathcal{A}$ has infinitely many orbits of size k for every $k \in \omega$, and $\mathcal{A}$ has infinitely many infinite orbits. Then there is a computable injection structure $\mathcal{B} = (B, g)$ such that $\mathcal{B}$ is isomorphic to $\mathcal{A}$ and $Fin(\mathcal{B})$ is of degree **c**.

Some Spectrum Results

- **Theorem** For any infiinite co-infinite c. e. set C and any c. e. character,
  (i) There is a computable injection structure $(\mathbb{N}, g)$ with $Rng(g) = C$ consisting of infinitely many orbits of type ω.
  (ii) If C is not simple, then there is a computable injection structure $(\mathbb{N}, h)$ with character K, with $Rng(h) \equiv_T C$, and with an arbitrary number of orbits of type $\mathbb{Z}$.
- **Theorem** For any c. e. set C and any computable injection structure $\mathcal{A}$ with infinitely many infinite orbits, there is a computable injection structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that C is $1-1$ reducible to an orbit of A.

Character versus Theory

- *Th*($\mathcal{A}$) is the first-order theory of $\mathcal{A}$
- *FTh*($\mathcal{A}$) is the elementary diagram of $\mathcal{A}$
  $\mathcal{A}$ is *decidable* if *FTh*($\mathcal{A}$) is computable.
- **Proposition** $\chi(\mathcal{A})$ is many-one reducible to *Th*($\mathcal{A}$).
  If *Th*($\mathcal{A}$) is decidable, then $\chi(\mathcal{A})$ is computable.

Computing $FTh(\mathcal{A})$

- For an injection structure $\mathcal{A} = (A, f)$,
  let $R^{\mathcal{A}}(n, a) \iff (\exists x) f^n(x) = a$
- **Theorem** $Fth(\mathcal{A}$ is computable from $R^{\mathcal{A}}$ together with $\mathcal{A}$.

  Sketch: Use quantifier elimination as in the theory of successor.
  First add the relations γ_n where $\gamma_n(a) \iff R^{\mathcal{A}}(a, n)$.
- **Theorem** For any $\mathcal{B}$, there exists $\mathcal{A}$ isomorphic to $\mathcal{B}$, such that $\mathcal{A}$ and $R^{\mathcal{A}}$ are computable from $\chi(\mathcal{A})$.

Decidability of $Th(\mathcal{A})$

- **Theorem** For any injection structure $\mathcal{A}$, $Th(\mathcal{A})$ and $\chi(\mathcal{A})$ have the same Turing degree.
  Thus $Th(\mathcal{A})$ is decidable if and only if $\chi(\mathcal{A})$ is computable.
- **Theorem** If $\chi(\mathcal{B})$ is computable, then there is a decidable $\mathcal{A}$ isomorphic to $\mathcal{B}$. (Hence $Th(\mathcal{B})$ is decidable.)
- **Corollary** If $\mathcal{A}$ has bounded character, then $Th(\mathcal{A})$ is decidable.

Decidability of Computably Categorical Structures

- **Proposition** There is a computably categorical injection structure $\mathcal{A}$ such that $Th(\mathcal{A})$ is not decidable.

  Sketch: Let W be a non-computable c.e. set and let $\mathcal{A}$ have character $\{(n, 1) : n \in W\}$ and no infinite orbits.

  This contrasts with the result for equivalence structures.
- **Proposition** For any computable character K, there is a decidable injection structure $\mathcal{A}$ with character K and with any number of orbits of types ω and Z. Furthermore, $\{a : \mathcal{O}_f(a) \text{ is finite}\}$ is computable.

  Sketch: There is a computable structure $\mathcal{B}$ with character K and thus there is a decidable structure $\mathcal{A}$ isomorphic to $\mathcal{B}$.
- **Corollary** If $\mathcal{B}$ has computable character and no infinite orbits, then $\mathcal{B}$ is decidable.

Being Injective

- **Theorem** $Inj = \{e : \mathcal{A}_e \text{ is an injection structure}\}$ and the set of indices of finitary injection structures are Π_2^0 complete sets.

The Infinite Orbits

- $Inj_m = \{e : \mathcal{A}_e$ structure with exactly m orbits of type $\omega\}$.
  $Inj_{\leq m} = \{e : \mathcal{A}_e$ structure with $\leq m$ orbits of type $\omega\}$
  and similarly define $Inj_{<m}$, $Inj_{>m}$ and $Inj_{\geq m}$.

  $Inj^n = \{e : \mathcal{A}_e$ structure with exactly n orbits of type $\mathbb{Z}\}$
  and similarly define $Inj^{\leq n}$, $Inj^{<n}$, $Inj^{>n}$ and $Inj^{\geq n}$.
  Combine these to define for example Inj_m^n to be
  $\{e : \mathcal{A}_e$ has m orbits of type ω and n orbits of type $\mathbb{Z}\}$.
- **Theorem** $Inj_{\leq m}$ is Π_2^0 complete, $Inj_{>m}$ is Σ_2^0 complete, and Inj_{m+1} is D_2^0 complete.

Proof Sketch

- Sketch: First define f so $\mathcal{A}_{f(e)}$ has all finite orbits if $e \in Inf$ and otherwise has one infinite orbit of type ω together with a finite number of finite orbits.
  ϕ^s with domain $\{0, 1, \ldots, s-1\}$ having some finite orbits of sizes $k_0, k_1, \ldots, k_s$, where s is the cardinality of $W_{e,s}$.
  $\phi^s(x) = x + 1$ except for $\phi^s(k_0 - 1) = 0$ and, for $i < s$, $\phi^s(k_0 + k_1 + \cdots + k_i - 1) = k_0 + k_1 + \cdots + k_{i-1}$.
  If a new element comes into $W_{e,s+1}$, let
  $\phi^{s+1}(s) = k_0 + k_1 + \cdots + k_{s-1}$, thus closing off the last orbit.
  Otherwise $\phi^{s+1}(s) = s + 1$.

Sketch Continued

- If W_e is finite and no new elements come in after stage s, then we have $\phi_{f(e)}(x) = x + 1$ for all $x > s$, and thus exactly one infinite orbit, of type ω.
  If W_e is infinite, then all orbits are finite.
- Reduce the D^0_2 complete set
  $D = \{\langle a, b\rangle : a \in Fin \,\&\, b \in Inf\}$.
  Let f and g be as above so that $e \in Fin$ if and only if $\mathcal{A}_{f(e)}$ has exactly one infinite orbit of type ω and all other orbits finite, and $e \in Inf$ if and only if all orbits of $\mathcal{A}_e$ are finite.
- Let $\mathcal{A}_{g(e)}$ be two copies of $\mathcal{A}_{f(e)}$. Now let $\mathcal{A}_{h(a,b)}$ consist of a copy of $\mathcal{A}_{f(a)}$ together with a copy of $\mathcal{A}_{g(b)}$.
  It can be checked that $\mathcal{A}_{h(a,b)}$ has exactly one infinite orbit of type ω if and only if $\langle a, b\rangle \in D$.

Orbits of Type $\mathbb{Z}$

- **Theorem** $Inj^{\leq n}$ is Π^0_3 complete, $Inj^{>n}$ is Σ^0_3 complete, and Inj^{n+1} is D^0_3 complete.
- Sketch: Reduce the Σ^0_3 complete set $Cof = \{e : W_e \text{ is cofinite}\}$ to Inj^1.
  Start to build ω chains going forward from each number $2m+1$ by mapping x to $2x$.
  When some m comes into $W_{e,s+1}$, find the longest sequence $k, k+1, \dots, m, m+1, \dots, n$ including m.
  Put the chains from $2(m+1)+1$ to $2n+1$ at the end of the $2m+1$ chain and fix them there.
  If $k < m$, put the newly expanded $2m+1$ chain at the end of the $2k+1$ chain.
  Add an element to the beginning of the $2k+1$ chain.

Sketch Continued

- If W_e is cofinite, there will be a least m such that every $n \geq m$ belongs to W_e.
  In that case, the orbit of $2m+1$ will be a $\mathbb{Z}$ chain, all of the $2n+1$ chains for $n > m$ will be included in this orbit, and there will be finitely many orbits of type ω for the numbers $k < m$.

Computable Categoricity

- **Theorem** The property of computable categoricity is Σ^0_3 complete (that is,
  $CCI = \{e : \mathcal{A}_e \text{ has finitely many infinite orbits}\}$
  is a Σ^0_3 complete set).
  Sketch: Reduce the Σ^0_3 complete set
  $Cof = \{e : W_e \text{ is cofinite}\}$.
  Define f such that for any e, $\mathcal{A}_{f(e)}$ will have finitely many infinite orbits if and only if W_e is cofinite.
  The orbits of $\mathcal{A}_{f(e)}$ will be exactly the orbits $\mathcal{O}(2i + 1)$ for $i \in \omega$ and the even numbers will be used in order to fill out the orbits.

Δ^0_2 Categoricity

- **Theorem** The property of Δ^0_2 categoricity is Σ^0_4 complete. Sketch: Fix a Π^0_4 set C and define a reduction f such that for any e, $\mathcal{A}_{f(e)}$ has only infinite orbits and has infinitely many orbits of type $\mathbb{Z}$ if and only if $e \in C$.

Σ^0_1 Structures

- Injection structures (A, f) where A is c.e. and f is the restriction to A of a partial computable function.

Complexity of the Orbits and Character

- Each infinite orbit is a Σ^0_1 set
- $\{(k, a) : a \in Rng(f^k)\}$ is a Σ^0_1 set
- $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ is a D^0_1 set, the intersection of a Π^0_1 set with A
- $\{a : \mathcal{O}_f(a) \text{ is infinite}\}$ is a Π^0_1 set
- $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$ is a Π^0_2 set
- $\{a : \mathcal{O}_f(a) \text{ has type } \omega\}$ is a Σ^0_2 set
- $\chi(\mathcal{A})$ is a Σ^0_1 set
- It follows that any Σ^0_1 injection structure is isomorphic to a computable structure

Isomorphisms

- **Theorem** For any Σ^0_1 injection structure $\mathcal{A}$, there exists a computable injection structure $\mathcal{B}$ and a computable isomorphism from $\mathcal{B}$ onto $\mathcal{A}$.
- **Theorem** If $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic Σ^0_1 injection structures with finitely many infinite orbits, then there is an isomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ such that both ψ and ψ^{-1} are partial computable.
- **Theorem** If $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic Σ^0_1 injection structures with either finitely many orbits of type $\mathbb{Z}$ or finitely many orbits of type ω, then there is a Δ^0_2 isomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_2$.
- **Theorem** If $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic Σ^0_1 injection structures, then there is a Δ^0_3 isomorphism $\psi : \mathcal{A}_1 \to \mathcal{A}_2$.

$Inf(\mathcal{A})$

- **Theorem** For any d.c.e. set B, there is a Σ^0_1 injection structure $\mathcal{A}$ such that B is $1-1$ reducible to $Inf(\mathcal{A})$.
- **Proof** Let $B = C - D$, where C and D are c.e. sets and $D \subset C$. Let $A = \{2n+1 : n \in C\} \cup \{2n : n \in \mathbb{N}\}$.
- For each n, we begin to define the orbit of $2n+1$ in $\mathcal{A}$ by setting $f(2n+1) = 2(2n+1)$, $f(2(2n+1)) = 4(2n+1)$ and so on, until we see that $n \in D$ at some stage $s+1$. Then let $f(2^s(2n+1)) = 2n+1$ and for $t > s$, let $f(2^t(2n+1)) = 2^{t+1}(2n+1)$.
  It follows that for each n, $n \in B$ IFF $2n \in Inf(A, f)$.

Complexity of Π_1^0 structures

- Each infinite orbit is a Σ_1^0 set
- $\{(k, a) : a \in Rng(f^k)\}$ is a Σ_2^0 set
- $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ is a Π_1^0 set,
- $\{a : \mathcal{O}_f(a) \text{ is finite}\}$ is a D_1^0 set, the intersection of A with a c.e. set
- $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$ is a Π_3^0 set
- $\{a : \mathcal{O}_f(a) \text{ has type } \omega\}$ is a Σ_3^0 set
- $\chi(\mathcal{A})$ is a Σ_2^0 set

Existence

- **Proposition** For any Σ^0_2 character K, which is infinite and coinfinite, there is a computable injection structure $\mathcal{A}$ with $\chi(A) = K$ and with any specified countable number of orbits of types ω and $\mathbb{Z}$.
- **Proposition** For any d.c.e. set B, there is a Π^0_1 injection structure $\mathcal{A}$ such that B is $1-1$ reducible to $Fin(\mathcal{A})$.

Categoricity

- **Theorem** If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic Π^0_1 injection structures with only finitely many infinite orbits, then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.

Infinite Orbits: Case One

- **Lemma** For any Π_1^0 injection structure $\mathcal{A}$, the relation $R(a, b)$, defined by $R(a, b)$ if and only if a and b are in the same orbit, is a D_2^0 set.
- **Theorem** If the Π_1^0 injection structure $\mathcal{A} = (A, f)$ has only finitely many orbits of type ω, then $\mathcal{A}$ is Δ_2^0 isomorphic to a computable structure.
- **Corollary** If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic Π_1^0 injection structures with only finitely many orbits of type ω, then $\mathcal{A}$ and $\mathcal{B}$ are Δ_2^0 isomorphic.

Infinite Orbits: Case Two

- **Proposition** For any infinite, co-infinite Σ^0_2 set C, there is a Π^0_1 injection structure $\mathcal{A} = (A, f)$ with all orbits of type ω such that $C \leq_T Ran(f)$.
- **Theorem** For any Σ^0_1 character K, there is a Π^0_1 injection structure $\mathcal{B}$ with character K, with infinitely many orbits of type ω and with an arbitrary number of orbits of type Z, such that $\mathcal{B}$ is not Δ^0_2 isomorphic to any Σ^0_1 injection structure.

Injection Structures in the Ershov Hierarchy

- $\mathcal{A} = (A, f)$ where A is an n-c.e. set and f is a computable function
- Each infinite orbit is a Σ_1^0 set
- $\{(k, a) : a \in Rng(f^k)\}$ is a Σ_2^0 set
- $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ is an n-c.e. set,
- $\{a : \mathcal{O}_f(a) \text{ is infinite}\}$ is the intersection of A with a Π_1^0 set, so is n-c.e. if n is even and $N + 1$-c.e. if n is odd.
- $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$ is a Π_3^0 set
- $\{a : \mathcal{O}_f(a) \text{ has type } \omega\}$ is a Σ_3^0 set
- $\chi(\mathcal{A})$ is a Σ_2^0 set

Back to Π^0_1 structures

- **Lemma** For any $n \in \mathbb{N}$ and any infinite *n*-c.e. set *B*, there is a Π^0_1 set *A* and a total computable $1-1$ function ϕ mapping *A* onto *B*.
- **Proposition** For any *n*-c.e. injection structure $\mathcal{A}$, there exist a Π^0_1 structure $\mathcal{B}$ and a computable injection $\phi : \mathbb{N} \to \mathbb{N}$ that maps $\mathcal{B}$ onto $\mathcal{A}$.
- **Corollary** If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic *n*-c.e. injection structures with only finitely many orbits of type ω, then $\mathcal{A}$ and $\mathcal{B}$ are Δ^0_2 isomorphic.

α-c.e. functions

- Let $g(x) = lim_s f(x, s)$, where f is a computable function.
  (i) g is an *n-c.e. function* if for all $x \in \omega$, $card(\{s : f(x, s) \neq f(x, s + 1)\}) < n$.
  (ii) g is an *ω-c.e. function* if there is a computable function g such that for all $x \in \omega$, $1 \leq card(\{s : f(x, s) \neq f(x, s + 1)\}) \leq g(x)$.
- **Proposition** For any nonempty Σ^0_2 set A there is a 2-c.e. function whose range is A.
- A function f is *graph-α-c.e.* if the graph of α is an α-c.e. set.
- **Proposition**
  (a) For every $n \in \omega$ there exists an $(n+1)$-c.e. function that is not graph-n-c.e.
  (b) There is a graph-2-c.e. function that is not an ω-c.e. function.

2-c.e. Categoricity: Case One

- **Theorem** There exist computable injection structures, each consisting of infinitely many orbits of type ω, which is not 2-c.e. isomorphic.
- Sketch: For a strucure $\mathcal{A} = (\omega, f)$, define the set $E^{\mathcal{A}}$ to be those elements of the form $f^{2n}(a)$ where $a \notin Ran(f)$. In the standard structure, $E^{\mathcal{A}}$ will be a computable set.
  We can build a computable copy in which $E^{\mathcal{A}}$ is n-c.e. or even ω-c.e. complete. Each orbit of $\mathcal{A}$ contains exactly one even number $2e$ and this orbit $\mathcal{O}(2e)$, will be used to defeat the eth ω-c.e. set C_e. That is, begin with $2e \notin Ran(f)$ and whenever e goes into or out of C_e, add an element to the beginning of $\mathcal{O}(2e)$, so that $2e \in E^{\mathcal{A}}$ IFF $e \notin C_e$.

2-c.e. Categoricity: Case Two

- **Theorem** There exist computable injection structures, each consisting of infinitely many orbits of type $\mathbb{Z}$, which is not ω-c.e. isomorphic.
- Sketch: Here we will diagonalize against the possible 2-c.e. isomorphisms h_e from the standard structure $\mathcal{A}$ to our structure $\mathcal{B}$, by having a pair of elements a_e and b_e in different orbits in $\mathcal{A}$ but having $h_e(a_e)$ in the same orbit with $h_e(b_e)$ in $\mathcal{B}$. We build $\mathcal{B}$ with infinitely many orbits of type $\mathbb{Z}$ by extending our finite orbits in both directions at each stage and by adding new orbits at each stage. When $h_e(a_e)$ and $h_e(b_e)$ are defined (or redefined) and we have them in different orbits, we simply combine those into one orbit.

2-c.e. Injections

- $\mathcal{A} = (\omega, f)$ where f is an n-c.e. function
- Each infinite orbit is a Δ^0_2 set
- $\{(k, a) : a \in Rng(f^k)\}$ is a Σ^0_2 set
- $\{(a, k) : card(\mathcal{O}_f(a)) \geq k\}$ is a Δ^0_2 set,
- $\{a : \mathcal{O}_f(a) \text{ is finite}\}$ is Σ^0_2
- $\{a : \mathcal{O}_f(a) \text{ has type } Z\}$ is a Π^0_3 set
- $\{a : \mathcal{O}_f(a) \text{ has type } \omega\}$ is a Σ^0_3 set
- $\chi(\mathcal{A})$ is a Σ^0_3 set
- So every 2-c.e. injection is isomorphic to a Π^0_1 injection

Existence

- **Theorem** Let K be a Σ_2^0 character.
  1. There is a 2-c.e. injection f such that (ω, f) has character K and has infinite orbits.
  2. If K possesses an s_1 function, then there is a 2-c.e. injection f such that (ω, f) has character K and has no infinite orbits.
- Sketch: Let $\mathcal{B} = (\omega, E)$ be a computable equivalence structure with character K (in the second case, $\mathcal{B}$ has infinitely many infinite orbits)
  Build f so that each equivalence class becomes an orbit
- **Question** Is the s_1 function necessary?

Future Work

- Structures (A, f) where f is finite-to-one.
- In particular 2 to 1 or $\leq$ 2 to 1.
- These are much more complicated.

- THANK YOU