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Injections, Orbits, and Complexity

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(joint work with Doug Cenzer and Jeff Remmel)

Consider an injection structure $\mathcal{A} = (A, f)$

- Have an injection $f : A \to A$. Given $a \in A$, the *orbit of* a (under f) is

$$\mathcal{O}_f(a) = \{b \in A : (\exists n \in \omega)[f^n(a) = b \ \vee \ f^n(b) = a]\}$$

- $Fin(\mathcal{A}) = \{a \in A : \mathcal{O}_f(a) \text{ is finite}\}$ is a Σ^0_1 (c.e.) set.
  $Inf(\mathcal{A}) = \{a \in A : \mathcal{O}_f(a) \text{ is infinite}\}$ is a Π^0_1 set.

  If $\mathcal{A}$ has only finitely many infinite orbits, then $Inf(\mathcal{A})$ is c.e., hence $Fin(\mathcal{A})$ is computable.

- The *character* $\chi(\mathcal{A})$ is defined by

$$\chi(\mathcal{A}) = \{(k, n) : 0 < k, n < \omega \ \& \ \mathcal{A} \text{ has at least } n \text{ orbits of size } k\}$$

  $\chi(\mathcal{A})$ is a Σ^0_1 set.

- Two types of *infinite orbits*:

  Z-orbits, which are isomorphic to $(\mathbb{Z}, S)$, in which every element is in $Ran(f)$;

  ω-orbits, which are isomorphic to (ω, S) and have the form $\mathcal{O}_f(a) = \{f^n(a) : n \in \omega\}$ for some $a \notin Ran(f)$.

- $\{a : \mathcal{O}_f(a)$ has type $Z\}$ is a Π^0_2 set.

  $\{a : \mathcal{O}_f(a)$ has type $\omega\}$ is a Σ^0_2 set.

- Hence, an injection structure is characterized by the number of orbits of size k for each finite k, and by the number of orbits of types Z and ω.

- $K \subseteq (\omega - \{0\}) \times (\omega - \{0\})$ is a *character* if for all $n > 0$ and k:

$$(k, n+1) \in K \Rightarrow (k, n) \in K$$

- For any Σ^0_1 character K,
  there is a computable injection structure $\mathcal{A} = (A, f)$
  with character K and with
  any specified finite or countable number of orbits of types ω and Z.

  Furthermore, $Fin(\mathcal{A})$ is computable and $Ran(f)$ is computable.

- If $\mathcal{A}$ is a computable injection structure with finitely many infinite orbits,
  then $\mathcal{A}$ is *relatively computably categorical*.

- Let $\mathbf{d}$ be a c.e. Turing degree.

  If $\mathcal{A}$ is a computable injection structure that has *infinitely many orbits of type* ω, then there is a computable injection structure $\mathcal{B} = (B, g)$ isomorphic to $\mathcal{A}$ such that $Ran(g)$ is a c.e. set of degree $\mathbf{d}$.

  If $\mathcal{A}$ is a computable injection structure that has *infinitely many infinite orbits of type* Z, then there is a computable injection structure $\mathcal{B} = (B, g)$ isomorphic to $\mathcal{A}$ in which $\mathcal{O}_g(b)$ is of type Z and is a c.e. set of degree $\mathbf{d}$.

- A computable injection structure $\mathcal{A}$ is computably categorical *iff*
  $\mathcal{A}$ has finitely many infinite orbits *iff*
  $\mathcal{A}$ is relatively computably categorical.

- Let C be a $\mathbf{\Sigma}^0_2$ set.
  There is a computable injection structure $\mathcal{A} = (A, f)$,
  with infinitely many orbits of type ω and infinitely many orbits of type Z,
  in which $\{a : \mathcal{O}_f(a)$ has type $\omega\}$ is
  a $\mathbf{\Sigma}^0_2$ set with Turing degree equal to $deg(C)$.

- A computable injection structure $\mathcal{A}$ is $\mathbf{\Delta}^0_2$ categorical *iff*
  $\mathcal{A}$ has finitely many orbits of type ω or finitely many orbits of type Z *iff*
  $\mathcal{A}$ is relatively $\mathbf{\Delta}^0_2$ categorical.

- (R. Miller) Let $\mathbf{d}$ be a Turing degree.
  A computable structure $\mathcal{A}$ is $\mathbf{d}$-*computably categorical* if
  for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$,
  there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{B}$ onto $\mathcal{A}$.

  The *degree of categoricity* of a computable structure $\mathcal{A}$ is
  the least Turing degree $\mathbf{d}$ for which $\mathcal{A}$ is $\mathbf{d}$-*computably categorical*.

- Let $\mathcal{M}$ be a computable $\mathbf{\Delta}_2^0$ categorical injection structure, which is not computably categorical. Then there is a computable injection structure $\mathcal{A}$ isomorphic to $\mathcal{M}$ such that the degree of categoricity of $\mathcal{A}$ is $\mathbf{0}'$.

  Let c.e. $C = \emptyset'$.
  First, consider the case when $\mathcal{M}$ has infinitely many orbits of type ω.
  Let $\mathcal{M}_0$ be the restriction of $\mathcal{M}$ to the orbits of type ω.

- Let $\mathcal{A}_0 = (A, f)$ be isomorphic to $\mathcal{M}_0$ and such that $Ran(f)$ is a computable set.

  Obtain a computable structure $\mathcal{B} = (B, g)$ isomorphic to $\mathcal{A}_0$ and an isomorphism $G : \mathcal{A}_0 \to \mathcal{B}$ such that

  $$(\forall i)[i \in \emptyset' \Leftrightarrow 2i + 1 \in Ran(g) \Leftrightarrow 2i + 1 \in G(Ran(f))].$$

  Since $Ran(f)$ is a computable set, we have $\emptyset' \leq_T G$ and hence $\mathbf{0}' \leq \deg(G)$.

- Any computable injection structure $\mathcal{A}$ is relatively Δ^0_3 categorical.

- Let $\mathcal{M}$ be a computable injection structure, which is not Δ^0_2 categorical. Then there is a computable injection structure $\mathcal{A}$ isomorphic to $\mathcal{M}$ such that the degree of categoricity of $\mathcal{A}$ is $\mathbf{0}''$.

- Let $C \subseteq \omega$ be a $\mathbf{\Sigma}_2^0$ set such that $deg(C) = \mathbf{0}''$.

  Without loss of generality, assume that $\mathcal{M}$ has no finite orbits.
  Then $\mathcal{M}$ has infinitely many orbits of type ω,
  and infinitely many orbits of type Z.

  Let $\mathcal{A}$ be a computable injection structure isomorphic to $\mathcal{M}$ such that A_ω,
  the set of all elements of $\mathcal{A}$ the orbits of which have type ω,
  is a computable set.

  We can obtain a computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ and
  an isomorphism $G : \mathcal{A} \rightarrow \mathcal{B}$ such that

  $$(\forall i)[i \in C \Leftrightarrow 2i + 1 \in G(A_\omega)].$$

  Since A_ω is a computable set, we have $C \leq_T G$ and thus $\mathbf{0}'' \leq \mathbf{deg}(G)$.
  Hence the degree of categoricity of $\mathcal{A}$ is $\mathbf{0}''$.

- Let $\mathbf{c}$ be a c.e. degree.

  Let $\mathcal{A}$ be a computable injection structure such that:
  $\mathcal{A}$ has infinitely many orbits of size k for every $k \in \omega$, and
  $\mathcal{A}$ has infinitely many infinite orbits.

  Then there is a computable injection structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that $Fin(\mathcal{B})$ is of degree $\mathbf{c}$.

- $Fin(\mathcal{A})$ cannot be a *simple* c.e. set.

  Each infinite orbit of $\mathcal{A}$ is a c.e. subset of the complement of $Fin(\mathcal{A})$.

- No infinite orbit of $\mathcal{A}$ can be a *simple* c.e. set.

  $Fin(\mathcal{A})$ is a c.e. subset of its complement.

- There is a non-simple c.e. set B such that for every computable injection structure $\mathcal{A}$:
  $B \neq Fin(\mathcal{A}) \wedge (B$ is not an orbit of $\mathcal{A})$.

- Let C be any infinite, coinfinite c.e. set.
  There is a computable injection structure $\mathcal{A} = (A, f)$,
  consisting of infinitely many orbits of type ω, such that $Ran(f) = C$.

- Let C be a non-simple c.e. set, and let K be any c.e. character.
  There is a computable injection structure $\mathcal{B} = (B, g)$ with character K,
  with $Ran(g) \equiv_T C$, and with any number of orbits of type Z.

- Let C be a c.e. set. Let $\mathcal{A}$ be a computable injection structure with infinitely many infinite orbits. There is a computable injection structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that C is $1-1$ reducible to an orbit of $\mathcal{A}$.

- An enumeration of structures $\mathcal{A}_e = (\omega, \phi_e)$, where ϕ_e is the eth partial computable function, includes every computable injection structure.

- $Inj = \{e : \mathcal{A}_e \text{ is an injection structure}\}$ is a Π^0_2 complete set.

  Inj is Π^0_2 : ($e \in Inj \iff \phi_e$ is total and $1-1$)

  For the completeness, consider $Tot = \{e : \phi_e \text{ is total}\}$ and define a reduction of Tot to Inj.

  Define $\phi_{g(e)}$ at stages s, beginning with $\phi_{g(e),0} = \emptyset$.
  If s is the least stage such that $\phi_{e,s+1}(x)\downarrow$,
  then if $\phi_{e,s+1}(x) \neq \phi_{g(e),s}(i)$ for all $i \neq x$ where defined,
  define $\phi_{g(e),s+1}(x) = \phi_{e,s+1}(x)$.
  If $\phi_{e,s+1}(x) = \phi_{g(e),s}(i)$ for some $i \neq x$,
  define $\phi_{g(e),s+1}(x) = n$ for the least $n \notin Ran(\phi_{g(e),s})$.

- Let $m > 0$. The set of indices of computable injection structures with no more than m orbits of type ω is Π^0_2 complete, and with more than m orbits of type ω is Σ^0_2 complete.

  $\mathcal{A}_e$ has more than m orbits of type ω if there exist at least $m+1$ elements x such that $x \notin Ran(\phi_e)$.

  Consider a Π^0_2 complete set $Inf = \{e : W_e \text{ is infinite}\}$.
  Define a computable function f so that:
  $e \in Inf$ iff $\mathcal{A}_{f(e)}$ has all orbits finite;
  $e \in Fin$ iff $\mathcal{A}_{f(e)}$ has one infinite orbit of type ω, and all other orbits finite.

  If W_e is finite and no new elements come in after stage s, then we have $\phi_{f(e)}(x) = x + 1$ for all $x > s$, and thus exactly one infinite orbit, and that of type ω.

  We then define $\phi_{g(e)}$ to consist of the disjoint union of m copies of $\phi_{f(e)}$.

- Let $m > 0$. The set of indices of computable injection structures with exactly m orbits of type ω is D_2^0 complete.

  A set is D_2^0 if it is a difference of two Σ_2^0 sets.

  We reduce the D_2^0 complete set
  $D = \{\langle i, j \rangle : i \in Fin \,\&\, j \in Inf\}$ to our index set.

  Let f be such that:
  $e \in Fin$ iff $\mathcal{A}_{f(e)}$ has exactly one infinite orbit of type ω;
  $e \in Inf$ iff all orbits of $\mathcal{A}_{f(e)}$ are finite.

  Let $\mathcal{A}_{g(e)}$ be two copies of $\mathcal{A}_{f(e)}$.
  Now let $\mathcal{A}_{h(i,j)}$ consist of a copy of $\mathcal{A}_{f(i)}$ together with a copy of $\mathcal{A}_{g(j)}$.
  $\mathcal{A}_{h(i,j)}$ has exactly one infinite orbit of type ω iff $\langle i, j \rangle \in D$.

- The set of indices of computable injection structures with no infinite orbits is Π_2^0 complete.

- Let $n > 0$. The set of indices of computable injection structures:
  with no more than n orbits of type Z is Π_3^0 complete;
  with more than n orbits of type Z is Σ_3^0 complete; and
  with exactly n orbits of type Z is D_3^0 complete.

  $\mathcal{A}_e$ has more than n orbits of type Z if
  there exist $n+1$ elements $x_0, \ldots, x_n$,
  each having an orbit of type Z, and no two being in the same orbit.

  For completeness, we reduce the Σ_3^0 complete set
  $Cof = \{e : W_e \text{ is cofinite}\}$
  to the set of indices of computable injection structures
  with one orbit of type Z.

- The property of computable categoricity
  for computable injection structures is Σ^0_3 complete.

  $\{e : \mathcal{A}_e$ is an injection structure with finitely many infinite orbits$\}$
  is a Σ^0_3 complete set.

- $\mathcal{A}_e$ has finitely many infinite orbits iff there exists a finite sequence $a_0, \ldots, a_{k-1}$ such that for every b, if $b \notin \mathcal{O}(a_i)$ for all $i < k$, then $\mathcal{O}(b)$ is finite.

  Define a reduction f such that for every e,
  $\mathcal{A}_{f(e)}$ has finitely many infinite orbits iff W_e is cofinite.

  The orbits of $\mathcal{A}_{f(e)}$ will be exactly the orbits $\mathcal{O}(2i+1)$ for $i \in \omega$, and the even numbers will be used to fill out the orbits.

  $\mathcal{O}_{f(e)}(2i+1)$ is finite iff $i \in W_e$.
  The function $\phi_{f(e)}$ is total and $1-1$.

- The property of Δ^0_2 categoricity
  for computable injection structures is Σ^0_4 complete.

- A computable injection structure $\mathcal{A}$ is Δ^0_2 categorical *iff*
  $\mathcal{A}$ has finitely many orbits of type ω or finitely many orbits of type Z.

- $I = \{e : \mathcal{A}_e$ is an injection structure with finitely many orbits of type ω
  or finitely many orbits of type $Z\}$
  is a Σ^0_4 complete set.

- Σ^0_3 condition: There exists a finite sequence $a_0, \ldots, a_{k-1}$ such that for every a, if $a \notin \mathcal{O}(a_i)$ for all $i < k$, then $\mathcal{O}(a)$ does not have type ω.

  Σ^0_4 condition: There exists a finite sequence $b_0, \ldots, b_{l-1}$ such that for every b, if $b \notin \mathcal{O}(b_i)$ for all $i < l$, then $\mathcal{O}(b)$ does not have type Z.

- For the completeness, let C be any Π^0_4 set.

  Then there is a Π^0_2 relation Q such that for every e:
  $e \in C$ iff $\{n : Q(e,n)\}$ is infinite.

  Now, there is a computable relation R such that for every n:
  $Q(e,n)$ iff $\{r : R(e,n,r)\}$ is infinite.

  Hence, $e \in C$ iff
  there are infinitely many n, such that there are infinitely many r,
  such that $R(e,n,r)$.

  Define a reduction f such that for every e, $\mathcal{A}_{f(e)}$ has only infinite orbits, and $\mathcal{A}_{f(e)}$ has infinitely many orbits of type Z iff $e \in C$.

  The orbits of $\mathcal{A}_{f(e)}$ will be exactly the orbits $\mathcal{O}(2i+1)$ for $i \in \omega$, and the even numbers will be used to fill out the orbits.

- We may assume that R is enumerated in stages.

  If (e, n, r) enters R at stage $s+1$,
  then we add a new element in front of $\mathcal{O}(2n+1)$.

  For every n, $\mathcal{O}_{f(e)}(2n+1)$ is infinite, and
  $\mathcal{O}_{f(e)}(2n+1)$ has type Z iff $\{r : R(e,n,r)\}$ is infinite.

  The function $\phi_{f(e)}$ is total and $1-1$.

$$f(e) \in I \iff e \notin C$$

  $\bar{C}$ is a $\mathbf{\Sigma}^0_4$ set.

THANK YOU!