Categoricity Properties for Computable Algebraic Fields

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Let A be a computable structure.

The *computable dimension* of A is the number of computable copies of A up to computable isomorphism.

A is *computably categorical* if it has computable dimension 1.

A is *relatively computably categorical* if every structure $B \cong A$ is isomorphic to A via a B-computable isomorphism.

In some classes, structural properties determine computable categoricity, e.g. algebraically closed fields, linear orders, Boolean algebras, Abelian groups.

In these classes, if a structure is not relatively computably categorical, it has computable dimension ω.

For each n, there are structures of computable dimension n.

There are computably categorical structures that are not relatively computably categorical.

Examples exists among partial orders, lattices, 2-step nilpotent groups, commutative semigroups, integral domains, etc.

A field is *algebraic* if it is an algebraic extension of its prime subfield ($\mathbb{Q}$ or $\mathbb{F}_p$).

A *splitting algorithm* for a computable field F is an algorithm deciding which elements of $F[X]$ factor properly in $F[X]$.

For a structure A, let B_A be the set of automorphic pairs of elements of A.

Thm (Miller and Shlapentokh). TFAE for a computable algebraic field F with a splitting algorithm.

1. B_F is computable.
2. F is computably categorical.
3. F is relatively computably categorical.

Let F be a computable field with prime subfield Q.

$x, y \in F$ are *conjugate* if they have the same minimal polynomial over Q.

$x, y \in F$ are *false conjugates* if they are conjugate but not automorphic.

View F as the union of a computable chain $Q = F_0 \subseteq F_1 \subseteq \cdots \subseteq F$ of finitely generated subfields.

Let $h(x)$ be the least s s.t. $x \in F_s$ and for every false conjugate y of x, there is no embedding of F_s into F mapping x to y.

We call h the *orbit function* of F w.r.t. the given chain.

Thm. TFAE for a computable algebraic field F.

1. F is relatively computably categorical.
2. The orbit function of F w.r.t. some computable chain is computably bounded.
3. The orbit function of F w.r.t. any given computable chain is computably bounded.

If A is relatively computably categorical then B_A is c.e.

Thm. There is a computably categorical field F s.t. B_F is not Σ^0_2.

Cor. There is a computably categorical field that is not relatively computably categorical.

Thm. Every computable algebraic field has computable dimension 1 or ω.

Pf. sketch. Let F be a computable algebraic field with prime subfield Q, viewed as the union of a computable chain $Q = F_0 \subseteq F_1 \subseteq \cdots \subseteq F$ of finitely generated subfields.

For each $s > 0$ we can effectively find a $z_s \in F_s$ that generates F_s and $p_s \in Q[X_0, \ldots, X_{s-1}]$ s.t. $p_s(z_0, \ldots, z_{s-1}, X)$ is the minimal polynomial of z_s over F_{s-1}.

Let $E \cong F$ be a computable field, with prime subfield identified with Q.

Let $I_{FE} = \{\sigma \in \omega^{<\omega} : \forall s < |\sigma| \, [p_s(\sigma(0), \ldots, \sigma(s)) = 0]\}$.

I_{FE} is a computable finitely branching tree whose paths correspond to the isomorphisms from F to E, in a degree-preserving way.

Thm (Goncharov). Let A and B be computable structures that are Δ^0_2 isomorphic but not computably isomorphic. Then A has computable dimension ω.

A function $f : \omega \to \omega$ is *leftmost-path approximable* if there is a computable l.o. $\prec$, computable functions g_0, g_1, and $k_0, k_1, \ldots \in \omega$ s.t. for all n,

1. k_n is $\prec$-least s.t. $\exists^\infty s \, \forall m \leqslant n \, [g_0(m, s) = k_m]$ and
2. $\forall^\infty s \, [g_0(n, s) = k_n \Rightarrow g_1(n, s) = f(n)]$.

The leftmost path of a computable finitely branching tree is leftmost-path approximable.

Thm (Hirschfeldt, Khoussainov, and Soare). Let A and B be computable structures that are leftmost-path approximably isomorphic but not computably isomorphic. Then A has computable dimension ω.

A computable algebraic field with a splitting algorithm is either relatively computably categorical or has computable dimension ω.

An algebraic field is either computably categorical or has computable dimension ω.

There are computably categorical algebraic fields that are not relatively computably categorical.

Open Question. Is there a field of finite computable dimension greater than 1?