Comparing classes of countable structures

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Outline

We want to compare "classification problems" for "nice classes" of countable structures.

1. What is a nice class?
2. What is the classification problem for a class?
3. How can we say in a precise way that the classification problem for one class is simpler than that for another class?

Conventions

1. Languages are computable
2. Structures have universe a subset of ω—sometimes all of ω.
3. Classes have a fixed language and are closed under isomorphism.

Nice classes

Let $K \subseteq Mod(L)$, where $Mod(L)$—L-structures with universe ω. Suppose K is closed under isomorphism.

Lopez-Escobar, 1965. K is Borel iff it is axiomatized by a sentence of $L_{\omega_1\omega}$.

Vaught, 1974. K is $\boldsymbol{\Sigma}_\alpha$ in the Borel hierarchy iff it is axiomatized by a Σ_α sentence of $L_{\omega_1\omega}$.

Vanden Boom, 2007. K is Σ_α in the effective Borel hierarchy iff it is axiomatized by a computable Σ_α sentence.

We will later relax the condition that the universe is all of ω.

Cardinality

We can compare classification problems using the cardinality of the set of isomorphism types. For example, the class *LO* of linear orderings has $2^{\aleph_0}$ isomorphism types, while the class *VS* of $\mathbb{Q}$-vector spaces has only $\aleph_0$. This provides a concrete way of saying why the classification problem for *LO* is harder than that for *VS*.

The class *LO* of linear orderings, the class *ApG* of Abelian p-groups, and the class *AF* of algebraic fields all $2^{\aleph_0}$ isomorphism types.

The classification problems seem different.

Borel cardinality

Definition (H. Friedman-Stanley, 1989).

1. A *Borel embedding* of K into K' is a Borel function $\Phi : K \to K'$ s.t. for $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$. We write $K \leq_B K'$ if there is such an embedding.

2. Classes K, K' have the same *Borel cardinality* if $K \equiv_B K'$; i.e., $K \leq_B K'$ and $K' \leq_B K$.

Distinctions we can and cannot make using Borel cardinality

Proposition (Friedman-Stanley, Anon.)

$$AF <_B ApG <_B LO$$

where LO—linear orderings, ApG—Abelian p-groups, AF—algebraic fields. .

Classes with $\aleph_0$ isomorphism types of countable structures all have the same Borel cardinality.

Effective cardinality

We change the setting, allowing structures with universe an arbitrary, possibly finite, subset of ω.

Definition (Calvert-Cummins-K-Quinn, 2004).

1. A *Turing computable embedding* of K into K' is a Turing operator $\Phi = \varphi_e$ s.t. for all $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A})$.
   We write $K \leq_{tc} K'$ if there is such an embedding.

2. Classes K, K' have the same *effective cardinality* if $K \leq_{tc} K'$ and $K' \leq_{tc} K$.

Pullback Theorem

Proposition (K-Quinn-Vanden Boom, 2007). Suppose $K \leq_{tc} K'$ via Φ. For any computable infinitary sentence φ in the language of K', we can effectively find a computable infinitary sentence φ^* in the language of K s.t. for $\mathcal{A} \in K$, $\mathcal{A} \models \varphi^*$ iff $\Phi(\mathcal{A}) \models \varphi$. Moreover, if φ is computable Σ_α, so is φ^*.

The proof uses forcing.

The structures here have universe an arbitrary subset of ω. This suggests a version of the effective Borel hierarchy, and an extension of Vanden Boom's result.

Some classes with $\aleph_0$ isomorphism types

Calvert-Cummins-K-Quinn, Safranski

$$FPF <_{tc} FLO \equiv_{tc} FUG \equiv NF$$

where *FPF*—finite prime fields, *FLO*—finite linear orderings, *FUG*—finite undirected graphs, *NF*—number fields.

For these classes, each element is determined by a single number. This is so even for *FUG*.

Application of Pullback Theorem

Proof that $FLO \not\leq_{tc} FPF$.

Suppose $FLO \leq_{tc} FPF$ via Φ.
Let $\mathcal{A}$ be a finite linear ordering, say with m elements, and suppose $\Phi(\mathcal{A})$ is a finite prime field with p elements.
Let φ be the sentence $\underbrace{1 + \ldots + 1}_{p} = 0$.

The pullback φ^* is computable Σ_1, a c.e. disjunction of finitary existential sentences, true in the ordering with m elements. It is also true in all larger linear orderings, a contradiction.

□

More classes with $\aleph_0$ isomorphism types

Proposition (Calvert-Cummins-K-Quinn, K-Quinn-Vanden Boom, K-Ocasio).

$$FLO <_{tc} VS \equiv_{tc} ACF <_{tc} FrG$$

where VS—$\mathbb{Q}$-vector spaces, ACF—algebraically closed fields of fixed characteristic, FrG—free groups.

Classes whose elements code a set

Proposition (van der Waerden, K). $DG <_{tc} AF$,
where AF—algebraic fields, DG—"daisy" graphs, with a "center", and a "petal" of length $2n + 3$ if $n \in S$ and $2n + 4$ if $n \notin S$.

Van der Waerden showed that $DG \leq_{tc} AF$.
The proof that $AF \not\leq_{tc} DG$ is another application of the Pullback Theorem. Suppose $AF \leq_{tc} DG$ via Φ. Take $\mathcal{A}, \mathcal{B}$, where $\mathcal{A}$ is isomorphic to $\mathbb{Q}$ and $\mathcal{B}$ has a square root of 2. There is an existential sentence φ true in $\Phi(\mathcal{A})$ and not true in $\Phi(\mathcal{B})$. The pullback φ^* is computable Σ_1. This is true in $\mathcal{A}$, and also in $\mathcal{B}$, a contradiction.

Structures determined by a family of sets

Proposition (Ocasio, 2011). $BDG \equiv_{tc} ARCF$, where
BDG—bunches of daisies, with a daisy graph for each set,
$ARCF$—Archimedean real closed ordered fields

The same is true, we think, for real closed ordered fields with certain specified value groups.

Some classes that lie on top

Proposition (H. Friedman-Stanley, Marker, Nies, Mal'tsev, Mekler). The following classes lie on top under both $\leq_B$ and $\leq_{tc}$.

1. directed graphs
2. undirected graphs
3. linear orderings
4. F_0—fields of characteristic 0
5. F_p—fields of characteristic p
6. 2-step nilpotent groups
7. real closed ordered fields

Other interesting classes

Proposition (K-Safranski, 2007). For any completion T of PA, $Mod(T)$ lies on top under $\leq_B$, but not under $\leq_{tc}$.

Proposition (H. Friedman-Stanley 1989, Fokina-K-Melnikov-Quinn-Safranski 2011). The class A_pG of Abelian p-groups does not lie on top under $\leq_B$, or $\leq_{tc}$.

To prove that $BDG \not\leq_{tc} A_pG$, we prove the following.

Proposition. Suppose K, K' are nice classes s.t. K has a pair of structures $\mathcal{A}, \mathcal{B}$ with the properties

1. $\mathcal{A} \ncong \mathcal{B}$,
2. $\mathcal{A}, \mathcal{B}$ satisfy the same computable infinitary sentences, and
3. $\omega_1^{\mathcal{A}} = \omega_1^{\mathcal{B}} = \omega_1^{CK}$,

and K' has no such pair of structures. Then $K \not\leq_{tc} K'$.