Interpreting Arithmetic in the Turing Degrees Below Genrics and Randoms

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The Setting

Our setting is that of the Turing degrees, $\mathcal{D}$.

Our overarching question is what are "most" degrees **x** like?

More specifically we are interested in analyzing the structure of $\mathcal{D}(\leq\mathbf{x})$ for "typical" degrees **x**.

“Most” Degrees

We consider the two standard notions of “most”..

- Category and
- Measure.

The “typical” degrees are then

- The generic degrees: the degrees of sets in comeager classes (of reals)
- The random degrees: the degrees of sets in classes (of reals) of measure 1.

Generics and Randoms

Definition

X is n-generic if for every Σ^0_n $S \subseteq 2^{<\omega}$ there is a $\sigma \subset X$ such that $\sigma \in S$ or $\forall \tau \supseteq \tau (\tau \notin S)$.

Definition

X is n-random if for every uniformly Σ^0_n collection V_k of open subsets of 2^ω of measure at most 2^{-k}, $X \notin \cap V_k$. (The V_k are specified by uniformly Σ^0_n subsets U_k of $2^{<\omega}$ such that $Z \in V_k \Leftrightarrow \exists \sigma \subset Z(\sigma \in U_k)$.)

Definition

X is generic (random) if it is n-generic (random) for every $n \in \mathbb{N}$.
A degree $\mathbf{x}$ is (n) generic or random if it contains a set which is (n) generic or random.

The Common Theories

Proposition

(Jockusch [80]) If $\mathbf{a}$ *and* $\mathbf{b}$ *are generic then* $\mathcal{D}(\leq \mathbf{a}) \equiv \mathcal{D}(\leq \mathbf{b})$ *so the degrees below all generics have the same elementary theory,* $Th(\leq G)$.

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Reason: The Turing degree of a set A, considered as a class of reals, is invariant under finite changes in its members. So either a simple forcing argument or an application of the $0-1$ law for category shows that for every sentence φ the degrees below $\mathbf{a}$ satisfy φ for comeager many $\mathbf{a}$ or satisfy $\neg\varphi$ for comeager many $\mathbf{a}$.

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The same argument using the $0-1$ law for measure shows that there is also a common theory $Th(\leq R)$ for the degrees below **a** for **a** random.

Questions

We want to address two natural questions. The first was raised long ago by Jockusch for category and explicitly just recently for measure by Barampalias, Day and Lewis. They also explicitly ask the second question.

- How much genericity (randomness) is needed to get to the common theory?
  More specifically, is there an *n* such that for every *n*-generic (random) $\mathbf{x}$, $Th(\mathcal{D} \leq \mathbf{x}) = Th(\leq G)$ $(Th(\leq R))$?

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  More specifically, is there an n such that for every n-generic (random) $\mathbf{x}$, $Th(\mathcal{D} \leq \mathbf{x}) = Th(\leq G)$ $(Th(\leq R))$?
- Are the two common theories the same?
  That is, does $Th(\leq G) = Th(\leq R)$?

Answers

We supply the expected negative answers for both questions.

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Interpreting arithmetic in degree structures and coding sets in the defined models.

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Distinguish between different levels of genericity (randomness).
That is, find properties, especially facts about $\mathcal{D}(\leq \mathbf{x})$, that vary as the level of genericity (randomness) of $\mathbf{x}$ changes.

Results

Theorem

There are sentences φ_n such that, for $n \geq 2$, $\mathcal{D}(\leq \mathbf{x}) \vDash \varphi_n$ for every $(n+1)$-generic (random) but such that $\mathcal{D}(\leq \mathbf{x}) \vDash \neg\varphi_n$ for some n-generics (randoms).

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Theorem

There is a sentence φ such that $\mathcal{D}(\leq \mathbf{x}) \vDash \varphi$ for every 3-random $\mathbf{x}$ but $\mathcal{D}(\leq \mathbf{x}) \vDash \neg\varphi$ for every 3-generic $\mathbf{x}$.

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We provide a coding scheme $\mathcal{S}(\overline{p})$ i.e. a set of formulas $\varphi_D(x,\overline{p}), \varphi_+(x,y,\overline{p}), \varphi_\times(x,y,\overline{p})$ and $\varphi_<(x,y,\overline{p})$ that provide, for each choice $\overline{\mathbf{p}}$ of parameters from $\mathcal{C}$, an interpretation of the language of arithmetic in $\mathcal{C}$ with $+, \times, \leq$ defined on $D = \{\mathbf{w} | \mathcal{C} \vDash \varphi_D(\mathbf{w})\}$ by the respective formulas to give a structure $\mathcal{M}(\overline{\mathbf{p}})$.

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We also have a correctness condition $\varphi_c(\overline{p})$ which guarantees (modulo $\mathcal{C}$ being sufficiently rich) that $\mathcal{M}(\overline{\mathbf{p}})$ is (isomorphic to) the standard model of arithmetic and an additional formula $\varphi_S(\overline{\mathbf{z}},\overline{\mathbf{p}})$ which codes sets in $\mathcal{M}(\overline{\mathbf{p}})$ as $\overline{\mathbf{z}}$ ranges over degrees in $\mathcal{C}$.

Crucial Properties

We need two important facts about our coding scheme.

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Thus when $\mathcal{D}(\leq \mathbf{x})$ is sufficiently rich the sets S such that both S and $\bar{S}$ are coded in some standard model below $\mathbf{x}$ are exactly the sets recursive in $\mathbf{x}''$.

Sufficiently Rich

There are various notions of richness which suffice and apply to different degree structures. The ones that apply to $\mathcal{D}(\leq \mathbf{x})$ for $\mathbf{x}$ at least 2-generic or 2-random are the following:

1. The 1-generic degrees are downward dense below $\mathbf{x}$. i.e. $\forall \mathbf{y} \leq \mathbf{x} \exists \mathbf{z} \leq \mathbf{y} (\mathbf{y}$ is 1-generic).

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We can now describe our sentences φ_n and φ.

The φ_n

φ_n says that there is a standard model of arithmetic $\mathcal{M}(\overline{\mathbf{p}})$ (as determined by our scheme $\mathcal{S}$ and correctness condition φ_c) and degrees $\overline{\mathbf{z}}$,$\overline{\mathbf{z}}'$ that code a set S and its complement $\bar{S}$, respectively, in $\mathcal{M}(\overline{\mathbf{p}})$ such that S is not recursive in $0^{(n)}$ (i.e. the formulation of this property in arithmetic as translated by our interpretation holds of the set S in $\mathcal{M}(\overline{\mathbf{p}})$).

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As no $(n+1)$-generic (random) is recursive in $0^{(n)}$, $\mathcal{D}(\leq \mathbf{x}) \vDash \varphi_n$ for every $(n+1)$-generic (random) and $n \geq 2$.

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As there are n-generic (random) $\mathbf{x} \leq \mathbf{0}^{(n)}$ and, for them $\mathbf{x}'' = \mathbf{x} \oplus \mathbf{0}'' \leq \mathbf{0}^{(n)}$ for $n \geq 2$ (Kautz for random), there are such $\mathbf{x}$ for which $\mathcal{D}(\leq \mathbf{x}) \vDash \neg\varphi_n$ for every $n \geq 2$.

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For $n = 1$ the same situation holds for the sentence φ_1 which says that there is no minimal degree. No 2-generic (random) bounds a minimal degree as above. Some 1-generics (randoms) do (Chong and Downey; Kumabe for generics; Barampalias, Day and Lewis for randoms).

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Clearly, if $\mathbf{x}$ is 3-random then $\mathcal{D}(\leq \mathbf{x}) \vDash \varphi$.

On the other hand, we claim that if $\mathbf{x}$ is 3-generic then $\mathcal{D}(\leq \mathbf{x}) \vDash \neg\varphi$.

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Corollary

If $\mathbf{x}$ *is 3-generic then* $\mathcal{D}(\leq \mathbf{x}) \vDash \neg\varphi$.

A Proof of the Proposition

As $X'' \equiv_T X \oplus 0''$ (Kautz), we only have to consider $R \leq X \oplus 0''_T$ so suppose $R = \Phi^{X \oplus 0''}$. We want to show that R is not 3-random. We define uniformly r.e. in $0''$ sets M_k and D_k of strings such that the M_k generate sets N_k of measure at most 2^{-k} and the D_k guarantee that $R \in N_k$.

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For each $n \geq 1$ and σ of length n we search for a $\tau \supseteq \sigma$ such that $\Phi^{\tau \oplus 0''} \upharpoonright 2n + k$ is defined. If there is one, we enumerate $\Phi^{\tau \oplus 0''} \upharpoonright 2n + k$ into M_k and τ into D_k for the first one found. Clearly, these sets are uniformly r.e. in $0''$. As $|\Phi^{\tau \oplus 0''} \upharpoonright 2n + k| = 2n + k$ the corresponding set put into N_k has measure 2^{-2n-k}. There are 2^n strings of length n and so the total contribution over these strings is at most 2^{-n-k}. Summing over all $n \geq 1$ we get that N_k has measure at most 2^{-k}.

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Now by the 3-genericity of X, there is a $\tau \subseteq X$ such that $\tau \in D_k$ or no extension of τ is in D_k. As $\Phi^{X \oplus 0''}$ is total, the former case must hold and so for some $\tau \subseteq X$, $\Phi^{\tau \oplus 0''} \upharpoonright m \in M_k$ and $\Phi^{\tau \oplus 0'' \upharpoonright m}$ is an initial segment of R for some m. Thus R is in the set N_k of measure at most 2^{-k} as required.

Effective Successor Models

We begin our coding of arithmetic with a specific effective form of coding ω orderings: nice effective successor structures (Shore [1981]).

All we need to know now is that the scheme provides a way of coding a sequence $\langle \mathbf{d}_n \rangle$ of independent degrees by parameters $\overline{\mathbf{q}}$ which generate a partial lattice including the $\mathbf{d}_n$. We assume that the first element $\overline{\mathbf{q}}_0$ of $\overline{\mathbf{q}}$ is a bound on all the other elements needed to determine this partial lattice.

Assuring the Crucial Properties

Theorem (Shore [1981])

Given a $\overline{\mathbf{q}}$ determining a nice effective successor structure, any set S such that $S = \{n | \mathbf{d}_n \leq \mathbf{g}_0, \mathbf{g}_1\}$ for any $\mathbf{g}_0, \mathbf{g}_1 \leq \mathbf{g}$ is $\Sigma_3^{G \oplus Q_0}$. If $\mathbf{q}_0 \leq \mathbf{a} < \mathbf{x}$, $\mathbf{x}$ is r.e. in $\mathbf{a}$ and $S \in \Sigma_3^X$ then there are $\mathbf{g}_0, \mathbf{g}_1 \leq \mathbf{x}$ such that $S = \{n | \mathbf{d}_n \leq \mathbf{g}_0, \mathbf{g}_1\}$.

Interpreting Arithmetic

We next extend the parameters $\overline{\mathbf{q}}$ to ones $\overline{\mathbf{p}}$ such that there are formulas with parameters $\overline{\mathbf{p}}$ providing an interpretation of arithmetic on the domain consisting of the $\mathbf{d}_n$ given by $\overline{\mathbf{q}}$ that identifies $\mathbf{d}_n$ with the nth element of this model, $\mathcal{M}(\overline{\mathbf{p}})$. That we can say this definably uses the way the $\mathbf{d}_n$ are generated by $\vee$ and $\wedge$ from $\overline{\mathbf{q}}$ in the nice effective successor structure and Slaman-Woodin forcing to define the set of the $\mathbf{d}_n$ and the relations on them giving a model of arithmetic.

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Downward density of the 1-generics suffices to provide enough comparison maps. We can guarantee that $\mathcal{M}(\overline{\mathbf{p}})$ is standard by saying that for every $\mathcal{M}(\overline{\mathbf{p}}')$ coded below $\mathbf{q}_0$ and every initial segment of $\mathcal{M}(\overline{\mathbf{p}})$ there is an definable order preserving map from initial segment of $\mathcal{M}(\overline{\mathbf{p}})$ onto an initial segment of $\mathcal{M}(\overline{\mathbf{p}}')$.

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The definitions of these finite maps is given by Slaman Woodin forcing. The technical facts we need about forcing below 1-generics are due to Greenberg and Montalbán [2004].

Below a 1-Generic

Theorem (Greenberg and Montalbán [2004])

For each recursive partial lattice $\mathcal{L}$, there is a recursive notion of forcing for which any 1-generic G computes an embedding of $\mathcal{L}$ into the degrees below G which is uniformly recursive in G. So one for which any 1-generic G compute degrees $\overline{\mathbf{q}}$ determining a nice effective successor structure in which the $\mathbf{d}_n$ are uniformly recursive in $\mathbf{q}_0$.

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Theorem (Greenberg and Montalbán [2004])

If $\mathbf{c}$ *uniformly bounds C_i and relations R_j on $\{\deg(C_i)\}$, there is a recursive notion of (Slaman-Woodin) forcing for which any* 1*-generic/C computes $\overline{\mathbf{p}}$ which code the relations R_j on $\deg(C_i)$ in the sense that there are fixed formulas φ_n (independent of C, C_i and R_j) such that, if R_j is of arity n, $R_j(\overline{\mathbf{z}}) \Leftrightarrow \varphi_n(\overline{\mathbf{z}},\overline{\mathbf{p}})$ and, moreover, $\varphi_n(\overline{\mathbf{z}},\overline{\mathbf{p}})$ holds if and only if it holds in any (equivalently all) ideals containing the degrees in $\overline{\mathbf{p}}$.*