The Mathematics of Relative Computability

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Channeling Alan Turing

Alan Turing had the remarkably prescient insight that understanding the means by which we work with things can be as important as, or even equivalent to, understanding those things. Equally remarkably, he combined a deep understanding of the abstract with pragmatic good sense.

Examples of Turing at Work

Famous:

Example

- Definition of computable via Turing Machine, universal machine, and non-computability of the Halting Problem and of first order validity.
  - *As we have heard, designed a machine to simulate the discrete moments in an algorithmic calculation.*
- Criteria for thinking machine
  - *Proposed a parallel between blindly distinguishing between male and female and blindly distinguishing between machine and human.*

Examples of Turing at Work

Less Famous:

Example

- (1938, thesis) Investigated hierarchies of proof principles along the ordinals.
    - *Self-described motivation, to circumvent the limitations of Gödel's Incompleteness Theorem by introducing intuition in a clear and systematic way.*

Examples of Turing at Work

Less Famous:

Example

- (1938, notes) Computable normal sequence, example taken from paper by Becher, Figueira, and Picchi.
  - *Gave an effective analysis of measure to convert from a measure-one argument to the computation of a particular instance.*
- (1950, letter) Questioning the model of information content in which computation costs nothing.
  - *Speculated on randomized algorithms with parallel architecture and concluded with a metaphoric reference to natural selection.*

Turing as Mathematician

Turing produced beautiful mathematics:

- Simple and unapologetic formulation
- Direct analysis, with sophistication as needed

Recursion Theory

I plan to present some of the mathematical developments that grew out of the initial investigations of the computable, in which Turing had a lead part.

How can we mathematically understand the means by which we define mathematical objects?

- Hierarchy of Definability
  - Some asides on the value added by having a precise calibration of the incomputable.
- Formalization, an alternative perspective
- Combination of the two

Recursion Theory

the hierarchy of definability and canonical models

Classifying the means to produce mathematical objects.

- Hierarchies of definability:
  - first order arithmetic
  - second order arithmetic
  - set theory
- Canonical models:
  - the natural numbers, with addition and multiplication, or equivalently the finite sets
  - the natural numbers with a collection of its subsets, such as recursive, arithmetic, hyperarithmetic
  - Gödel's universe of constructible sets and its generalizations to inner models for large cardinals

The Arithmetic Hierarchy

Definition

The arithmetically definable subsets of the natural numbers are those generated from $\{\emptyset\}$ as follows.

- If X is arithmetically definable and Y is recursive relative to X, then Y is arithmetically definable.
- If X is arithmetically definable then so is X', the universal recursively enumerable in X set.

The arithmetically definable sets are exactly those definable in first order arithmetic with addition and multiplication. They appear in a natural ω-length hierarchy measured by the number of alterations of quantifiers in their definitions.

Definability and Randomness

an aside

There are intuitively-clear mathematical concepts best expressed in the language of definability.

Example

How can we talk of a particular object as being *random*?

For a given form of definability Γ, we say that R is Γ-random if R does not belong to any measure-zero set defined in Γ.

Thus, we can talk about particular elements of 2^ω or particular curves as being random binary sequences for a given probability measure or random sample paths for a Brownian motion.

Beginning the Transfinite Hierarchy

Definition (work of Church, Davis, Gandy, Kleene, etc.)

The hyperarithmetically definable subsets of the natural numbers are those generated with this additional clause.

- If $\prec$ is a recursive well-ordering of ω then the following sequence $(J_e : e \in \omega)$ is hyperarithmetic, where we use parentheses to indicate recursive join.
  - If z is the $\prec$-least number, then $J_z = \emptyset$.
  - If p is the $\prec$-immediate predecessor of s, then $J_p' = J_s$.
  - If ℓ is a $\prec$-limit, then $J_\ell = (J_n : n \prec \ell)$.

A beautiful theorem of Kleene asserts that the hyperarithmetic sets are exactly those which are Δ^1_1-definable.

The Transfinite Hierarchy

- Jensen showed how to extend this hierarchy, based on the existential first order quantifier, through the ordinals that are countable in Gödel's Universe of Constructible Sets.
- Under additional meta-mathematical set-theoretic assumptions, it goes even further.

Question

Is this the calibration of definability, at least for subsets of ω?

The Turing Degrees

The partial ordering of the Turing degrees is the standard algebraic/structural representation of relative definability.

- A Turing degree is the equivalence class of a subset of ω under equi-computability
- The Turing degrees are ordered by relative computability

For example, the recursive sets form the least degree and the arithmetic sets form a natural ideal.

A Fundamental Question

Sacks, Martin

Characterize those operations on real numbers which are invariant under equi-definability, such as the Turing jump $X \mapsto X'$ or the function mapping X to the set of reals which are arithmetically definable from X.

Excluding applications of the Axiom of Choice, all the known non-trivial examples come from notions of relative definability.

- Degree invariant functions from reals to reals come from universal sets.
- Degree invariant functions from reals X to sets of reals containing X come from closures under relative definability.

Degree Invariant Operations

Question (Sacks)

Is there an e such that the function $X \mapsto W_e^X$ satisfies the following conditions?

- *For all X, $X <_T W_e^X <_T X'$.*
- *For all X and Y, if $X \equiv_T Y$ then $W_e^X \equiv_T W_e^Y$.*

Remark

Sacks posed this question in anticipation of an interesting construction.

Martin Measure

Definition

1. A *cone of reals* is a set $\{X : X \geq_T B\}$, for some base B.
2. A property P on the Turing degrees, D, *contains a cone* iff there is a cone of reals all of whose degrees satisfy P.

Theorem (Assuming the Axiom of Determinacy)

Suppose a set $A \subseteq 2^\omega$ is closed under $\equiv_T$. Then one of A or $2^\omega \setminus A$ contains a cone, i.e. the cone filter is a $\{0,1\}$-valued measure.

For degree-invariant functions, we define order preserving on a cone, constant on a cone, and other notions, similarly. We define $F \geq_M G$ iff $F(X) \geq_T G(X)$ on a cone.

Martin's Conjecture (MC)

Conjecture (Martin)

Assume ZF+AD+DC.

I. *If F is degree invariant and not increasing on a cone, then F is constant on a cone.*

II. *$\leq_M$ is a prewellordering of the set of degree invariant functions which are increasing on a cone. Further, if f has $\leq_M$-rank α, then f' has $\leq_M$-rank $\alpha + 1$, where $f' : x \mapsto f(x)'$ for all x.*

Remark

A similar conjecture (MC-sets) applies to functions F mapping reals to sets X of reals $F(X)$ such that $X \in F(X)$ and $X \leq_T Y \rightarrow F(X) \subseteq F(Y)$.

Martin's Conjecture

Supporting evidence:

- (Slaman-Steel)
  - MC is true for $<_T$-decreasing functions.
  - MC is true for Borel order-preserving functions.
  - MC is true for uniformly-invariant functions, such as those which come from universal sets for notions of relative definability.
- (Slaman)
  - MC-sets is true if $Y \in F(X)$ is a Borel property.

Return to Sacks's Question

an aside

MC asserts the following about Sacks's question.

(∗) For any e such that $X \mapsto W_e^X$ is degree-invariant, there is a B such that either

- $(\forall X \geq_T B)\left[W_e^X + X \equiv_T X\right]$ or
- $(\forall X \geq_T B)\left[W_e^X + X \equiv_T X'\right]$.

The validity of MC rests upon the fullness of the set of real numbers, as reflected by definable instances of the axiom of determinacy.

Proposition (Bienvenu, Lafitte, and Slaman)

(∗) fails when restricted to the hyperarithmetic sets.

Connections with the theory of Borel Equivalence Relations

an aside

There is an intimate connection between these questions and the structure theory for Borel equivalence relations.

Andrew Marks, In his PhD thesis, gives a variety of Borel combinatorial principles related to, in some cases equivalent to, the existence of Borel degree-invariant functions.

Martin's Conjecture

Martin's Conjecture is a particular realization of the view that all notions of relative definability extending relative computability appear in the logical hierarchy based on first order quantification over the finite sets.

By results already known, there are severe limitations on possible alternate notions of "relatively definable."

The Axiomatic Hierarchy

Parallel to the hierarchy of definability is the hierarchy of axiomatic theories, which formalize the basic properties of the canonical models.

- Axiomatic systems
  - Peano Arithmetic and its subsystems, such as $B\Sigma_n$, $I\Sigma_n$
  - Second Order Arithmetic and its subsystems, such as RCA_0, WKL_0 (a compactness principle), ACA_0, Π^1_1-CA_0
  - Zermelo-Fraenkel set theory, $0^\#$, measureables, supercompacts

These systems approximate the theories of their standard models.

Definability vs Provability

which tells us more about the nature of mathematical investigations?

The hierarchy of definability and the hierarchy of axiom systems within set theory are parallel and entwined attempts to quantitatively and systematically describe the ingredients of mathematical investigations.

They provide alternate means to answer questions like the following.

- Whether there is an object, such as a real number, which can be produced using methods, principles, techniques of Type A and which satisfies Property B
- Whether principles of Type A and be used to settle questions of Type B

A Recursion Theorist's Assumption

At least with respect to familiar principles, both sorts of questions can be formulated and settled by directly considering the nature of "Type A," as Turing did with the nature of computation, with minimal reliance on the formalization of theories.

Reverse Mathematics

Reverse Mathematics is the program set forth by Friedman and Simpson to investigate the following question.

Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?

The reverse mathematical approach to the problem is two-fold.

- Frame the question in formal subsystems of second-order arithmetic.
- Locate the theorems of ordinary, non-set-theoretic mathematics within these subsystems. In a substantial number of cases, the theorems are formally equivalent to the subsystem in which they can be proven (over a weaker subsystem RCA_0).

Reverse Math vs. Recursion Theorist's Assumption

Caveat: there may not be universal agreement with this view

The typical reverse mathematics question asks whether RCA_0 can prove

$$\forall X_1 \exists Y_1 \psi_1 \to \forall X_2 \exists Y_2 \psi_2,$$

where ψ_1 and ψ_2 are arithmetic formulas. Typically, $\forall X_1 \exists Y_1 \psi_1$ is formulated in the language of ordinary, non-set-theoretic mathematics and $\forall X_2 \exists Y_2 \psi_2$ is formulated as a principle of logic.

Understanding $\forall X_1 \exists Y_1 \psi_1 \rightarrow \forall X_2 \exists Y_2 \psi_2$

Caveat: there may not be universal agreement with this view

The typical solution to a reverse mathematics question is one of two types.

- (Reversal) Show that for every X_2 there is an X_1 recursive in X_2, such that for any Y_1 satisfying $\psi_1(X_1, Y_1)$ there is a Y_2 recursive in Y_1 for which $\psi_2(X_2, Y_2)$. Typically, the proof is by (ingenious) translation from ψ_2 to ψ_1 and can be formalized in RCA_0.
- (Non-reversal) Show that there is an ideal I in the Turing degrees as follows.
  - For every $X_1 \in I$ there is a $Y_1 \in I$ such that $\psi_1(X_1, Y_1)$
  - There is an $X_2 \in I$ such that for all $Y_2 \in I$, $\neg\psi_2(X_2, Y_2)$. Typically, X_2 is recursive.

If one only works with ω-models, then the formalism is not needed.

Challenging the Recursion Theorist's Assumption

Question

What are the finitary consequences of infinitary principles?

Here, one can ask about principles such as the existence of an infinite random source, infinite combinatorial principles such as Ramsey's Theorem, or set theoretic principles such as the existence of infinitely many cardinals or large cardinals, as studied by H. Friedman.

The Recursion Theorist takes arithmetic on $\mathbb{N}$ as given and cannot argue semantically about how its theory is affected by infinitary principles.

Definability Theoretic Thinking in Non-ω-models

Our understanding of the fundamentals of definability applies perfectly well in non-standard models.

- Understanding the jump in the Jockusch-Soare Low Basis Theorem applies to conclude Harrington's Theorem that WKL_0 is conservative over RCA_0 for Π^1_1-statements.
- Understanding the double jump in Ramsey's Theorem for Pairs, applies to conclude Cholak-Jockusch-Slaman's theorem that RT^2_2 is conservative over $RCA_0 + I\Sigma_2$ for Π^1_1-statements.

However, there are mysteries here, too. It is open to exactly characterize the first-order consequences of RT^2_2 or of the existence of random reals.

Missing Ingredients

We have two powerful tools with which to analyze relative definability.

- The hierarchy of definability, based on the Turing jump, provides the ability to calibrate relative definability.
- Forcing, interpreted broadly to include priority constructions and other effective implementations, provides the ability to approximate arbitrarily complicated definitions relative to G while constructing G.

To have a widely applicable technology to answer questions about infinite/finite, we need a third set of tools.

We need tools to fine-tune the underlying structure of arithmetic so as to control the behavior of the hierarchy of definability built upon it.

A Final Thought

We who work in Mathematical Logic are attempting to understand the interaction between the mathematical objects and the means needed to speak about them. This is as fundamental an investigation as any other in Mathematics.

No one's contribution to this investigation is greater than Alan Turing's.