Classifying Computably Enumerable Equivalence Relations

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The Incomputable

The Incomputable, Newport Pagnell, 11-16 June 2012

Object of study

Positive (or, computably enumerable) equivalence relations (or, *ceers*) on the set ω of natural numbers, together with reducibility $\leq$:

Definition

$R \leq S$ if there exists a computable function f such that

$$x\,R\,y \Leftrightarrow f(x)\,S\,f(y).$$

[Maltsev 1965], [Ershov 1971], [Ershov 1973], [Ershov 1975]

My interest in ceers was originally motivated by the following problem:
Is it true that for every positive equivalence relation R there exists a computable function f such that, for all numbers x, y,

$$xRy \Leftrightarrow \vdash_{PA} f(x) \leftrightarrow f(y)?$$

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The answer turned out (Bernardi and S. 1983) to be "Yes", in fact for every such R there exists a Σ_1 formula $\varphi(v)$ such that for all numbers x, y,

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(Thus, extending, classical representation results of c.e. sets in strong enough formal theories)

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Further later on (Montagna and S. 1984), it even turned out that for every c.e. preorder $\leq$ there exists a Σ_1 formula $\varphi(v)$ such that for all numbers x, y,

$$x \leq y \Leftrightarrow \vdash_{PA} \varphi(\overline{x}) \rightarrow \varphi(\overline{y}).$$

The category *Eq* of equivalence relations

- *Objects of Eq*: Equivalence relations on ω.
- *Morphisms from R_1 to R_2*: functions $\pi : \omega/_{R_1} \longrightarrow \omega/_{R_2}$ such that there is a computable function f satisfying:

$$\pi([x]_{R_1}) = [f(x)]_{R_2} :$$

so morphisms are induced by computable functions satisfying:

$$x\, R_1\, y \Rightarrow f(x)\, R_2\, f(y).$$

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Notions of *monomorphism*, *epimorphism*, *isomorphism* etc. are given in the obvious ways.
Eq is just another way of looking at the *category of numberings*, a popular research topic in Novosibirsk.

The category Eq^P of positive equivalence relations

Eq^P is the full subcategory of Eq, whose objects are ceers.

monomorphisms=injective
epimorphisms=surjective

Thus

$R \leq S$ iff there is a mono from R to S iff R is a *subobject* of S.

Useful notion is also that of a R being a *quotient object* of S: if there is an epi from S to R.

Examples of positive equivalence relations

- Id_n, with n equivalence classes ($n \geq 1$); Id =identity.
- every ceer is of the form R_φ (for some partial computable function φ) [Ershov 1971] where

  $$xR_\varphi y \Leftrightarrow (\exists m, n)[\varphi^m(x)\downarrow = \varphi^n(y)\downarrow];$$

  (This also suggests a possible way to index ceers).

- $x \sim_T y$ where $x = \ulcorner \sigma \urcorner, y = \ulcorner \tau \urcorner$ and $T \vdash \sigma \leftrightarrow \tau$, where, say, $T = PA$, ($\ulcorner \_ \urcorner$ bijective, effective, etc.);
- For every $n \geq 1$, identify (via a suitable Gödel numbering $\ulcorner \_ \urcorner_n$) Σ_n-sentences of T and natural numbers and define:

  $$x \sim_n y \Leftrightarrow \vdash_T \sigma \leftrightarrow \tau,$$

  where $x = \ulcorner \sigma \urcorner_n$, and $y = \ulcorner \tau \urcorner_n$.
- *Unidimensional* equivalence relations are of the form R_A, where A is c.e. and

  $$x \, R_A \, y \Leftrightarrow [x, y \in A \text{ or } x = y]$$

Degrees of positive equivalence relations

Let $R \equiv S$ if $R \leq S$ and $S \leq R$. Denote by $\deg(R)$ the $\equiv$-equivalence class of R, and let

$$\deg(R) \leq \deg(S) \Leftrightarrow^{df} R \leq S.$$

$\mathcal{P} = \langle ob(Eq^P), \leq \rangle$ denotes the poset of degrees of ceers.

The poset $\mathcal{P}$

Theorem

$\mathcal{P}$ is a bounded poset.

Definition

Ceers in the top element are called *universal*.

Lemma

The following hold:

- If A, B are c.e. sets, with B infinite, then

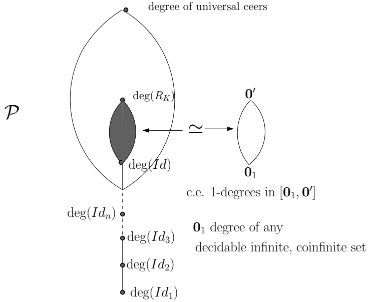
$$A \leq_1 B \Leftrightarrow R_A \leq R_B.$$

- If $Id \leq R \leq R_A$ then there exists a c.e. B such that $R \equiv R_B$, some $B \leq_1 A$.

Lemma

If A is simple, then $Id \not\leq R_A$.

A look at $\mathcal{P}$



Corollary

We have $[\deg Id, \deg R_K] \simeq [\mathbf{0}_1, \mathbf{0}']$. *Thus* $\mathcal{P}$ *is neither an upper semilattice nor a lower semilattice (It follows from [Young 1963]).*

Corollary

$[\mathbf{0}_1, \mathbf{0}']$ *is elementarily definable in* $\mathcal{P}$ *with parameters* $\deg(R)$ *and* $\deg(R_K)$. *Hence the first order theory of* $\mathcal{P}$ *is undecidable.*

A jump operation on ceers

Given a ceer R, define (Gao and Gerdes 2001)

$$xR'y \Leftrightarrow x = y \text{ or } \varphi_x(x)\downarrow\, R\, \varphi_y(y)\downarrow$$

Then

- $R \leq R'$;
- $R \leq S \Leftrightarrow R' \leq S'$;
- if R is not universal then R' is not universal.

Remark

- $Id_1' = R_K$;
- Id' is the ceer yielding the partition $\{K_i : i \in \omega\} \cup \{\{x\} : x \notin K\}$, where $K_i = \{x : \varphi_x(x)\downarrow = i\}$.

Does the jump operation have fixed points besides the degree of universal ceers?

Does the jump operation have fixed points besides the degree of universal ceers?
The answer is no:

Theorem

For every ceer E, we have $E' \leq E$ iff E is universal.

Corollary

The poset of degrees of ceers is upwards dense.

Idea of proof

It uses infinitely many indices that we control by the Recursion Theorem.
(Formal justification is provided for instance by the Case Functional Recursion Theorem:

Lemma (Case Functional Recursion Theorem)

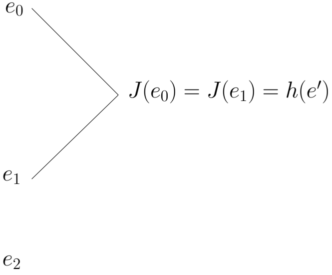
Given a partial computable functional F, there is a 1-1 total computable function f such that, for every e, x,

$$F(f, e, x) = \varphi_{f(e)}(x).$$

Let $E' \leq E$ via h, and let R be any ceer.
Use ∞-many indices that we control by the Recursion Theorem: $e_0, e_1, \ldots$ (and infinitely many more).
At the end define $f(i) = h(e_{i)}$, and show

$$i \, R \, j \Leftrightarrow e_i \, E' \, e_j (\Leftrightarrow h(e_i) \, E \, h(e_j)).$$

Idea of Proof: Write $J(e) = \varphi_e(e)$.



Notice,

$$\begin{aligned} J(e') \downarrow = J(e'') \downarrow &\Rightarrow e' \, E' \, e'' \\ &\Rightarrow h(e') \, E \, h(e'') \\ &\Rightarrow e_1 \, E' \, e_2. \end{aligned}$$

Etc.

Care must be taken (by carefully controlling convergence of various $J(e)$) to collapse only what we *need* to collapse.

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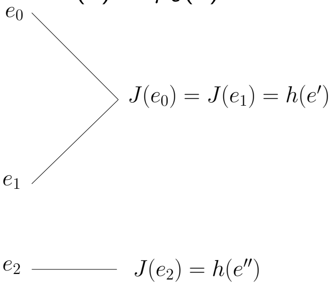

Notice,

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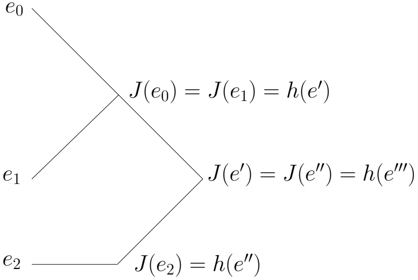

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Etc.

Care must be taken (by carefully controlling convergence of various $J(e)$) to collapse only what we *need* to collapse.

Prt II: Precomplete equivalence relations Malt'sev 1963, Ershov 1973]

Definition

An equivalence relation R is *precomplete* if for every partial computable function φ there exists a total computable function f such that for all n,

$$\varphi(n)\downarrow \Rightarrow \varphi(n)\, R\, f(n).$$

In the above cases we say that f *makes* φ *total modulo* R.

Theorem (Fixed Point Theorem)

R is precomplete if and only if there is a computable function fix *such that, for every n,*

$$\varphi_n(\mathrm{fix}(n))\downarrow \Rightarrow \varphi_n(\mathrm{fix}(n))\; R\; \mathrm{fix}(n).$$

Examples of precomplete ceers

- R_u, where u is a universal partial computable function [Malt'sev 1965];
- ([Visser 1980]) Recall that for every $n \geq 1$, PA has a Σ_n-truth predicate $T_n(v)$, i.e. a Σ_n formula such that for all Σ_n-sentences σ

$$\vdash_T \sigma \leftrightarrow T_n(\overline{\ulcorner \sigma \urcorner})$$

  (identify numbers with Σ_n sentences, via a suitable Gödel numbering $\ulcorner \_ \urcorner_n$.)
  Then $\sim_n$ is a precomplete ceer. (Given partial computable φ, take -more or less- $f(m) = \ulcorner \exists v(\varphi(m) = v \wedge T_n(v)) \urcorner_n$).
  (From this it will follow that all ceers are "represented" by Σ_1 formulas.)
- Provable equality $\sim_{\lambda\beta}$ of $\lambda\beta$-calculus:

$$\ulcorner M \urcorner \sim_{\lambda\beta} \ulcorner N \urcorner \Leftrightarrow^{df} M =^{\beta} N.$$

Precomplete vs universality

Theorem (Bernardi and S., 1983)

Every precomplete ceer is universal.

Theorem

$\sim_T$ *is universal but not precomplete.*

- $\sim_T$ is not precomplete since it does not satisfy the Fixed Point Theorem of precomplete equivalence relations: The computable function $\neg$ has **no** fixed point by consistency of T.
- universality follows from the fact that for every $n \geq 1$, $\sim_n \leq \sim_T$.

Corollary

There are non isomorphic universal ceers. So failure of the Myhill Isomorphism Theorem for universal ceers!

Uniformly finitely precomplete (ufp) numberings

Notice that, although not precomplete, $\sim_T$ is "locally" precomplete, i.e. every partial computable function with finite range can be totalized modulo $\sim_T$ since there is some $n \geq 1$ such that all sentences in the range of φ are Σ_n, and thus we can totalize modulo $\sim_n$. This leads to the following definition (we directly give the "uniform" version):

Definition (Montagna 1982)

R is *uniformly finitely precomplete* (notation: *ufp*) if there is a total computable function f, such that, for all e, D, n, (where D is a finite set)

$$\varphi_e(n) \downarrow \in [D]_R \Rightarrow \varphi_e(n) R f(e, D, n).$$

Notice, if R is ufp, then we can always totalize partial computable functions with finite ranges.

Examples

- Every precomplete equivalence relation is ufp.
- $\sim_T$ is ufp. For the proof use the fact that, given φ and D, all sentences in D fall into some finite level Σ_n.

Theorem (Montagna 1982)

Every ufp ceer is universal.

e-complete equivalence relations [Montagna 1982], [Lachlan 1987]

So inside the class of ufp ceers, we have the precomplete ceers (which satisfy the Ershov Fixed Point Theorem), and at the other extreme we have ceers with a total *diagonal* function, i.e. a total computable function f with no fixed point modulo the equivalence!

Definition (Montagna 1982; Bernardi and Montagna 1984)

An equivalence relation R is *e-complete* if it is ufp and it has a total diagonal function.

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An equivalence relation R is *e-complete* if it is ufp and it has a total diagonal function.

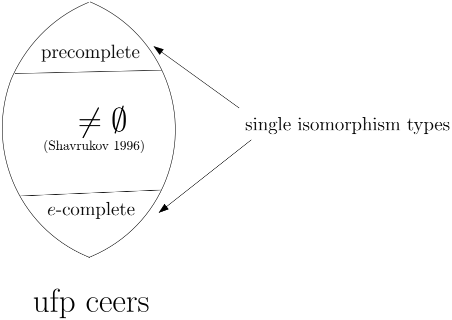
So, $\sim_T$ is e-complete.

Theorem (Isomorphism theorems)

The following hold:

- *(Lachlan 1987) All precomplete ceers are isomorphic;*
- *(Montagna 1982) All e-complete ceers are isomorphic.*

Summary so far: the ufp ceers



- The precomplete ceers are all isomorphic to $\sim_1$
- The *e*-complete ceers are all isomorphic to $\sim_T$.
- The ufp that are not *e*-complete are *weakly precomplete* in the sense of Badaev 1991.

Theorem (Bernardi and Montagna 1984)

A ceer R is ufp iff R is a quotient of $\sim_T$, i.e. $R \supseteq \sim_T$ up to isomorphisms.

Partitions of ω into effectively inseparable sets

Recall:

Definition

Two disjoint c.e. sets A and B are *effectively inseparable* if there is a computable function p (called *productive*) such that, for all pairs u, v,

$$A \subseteq W_u \text{ and } B \subseteq W_v \text{ and } W_u \cap W_v = \emptyset \Rightarrow p(u, v) \notin W_u \cup W_v.$$

Definition

A ceer R is:

- *effectively inseparable* (abbreviated, *e.i.*) if it yields a partition of ω into effectively inseparable sets;
- *uniformly effectively inseparable* (abbreviated, *u.e.i.*) if it is e.i. and there is a uniform productive function, i.e. a computable function $g(a, b)$ such that if $[a]_R \cap [b]_R = \emptyset$ then $\varphi_{g(a,b)}(u, v)$ is a productive function for the pair $[a]_R, [b]_R$.

ufp ceers are u.e.i.

Theorem (Visser 1980, Bernardi 1981, Bernardi and S. 1983, Montagna 1982)

Every ufp ceer is u.e.i.

In view of this, does the property of being u.e.i. imply universality?

The following theorem subsides all universality results seen so far, and is a natural companion of [Myhill 1955], [Smullyan 1961], and [Cleave 1961]

Theorem

Every u.e.i. ceer is universal.

Recall, for instance [Smullyan 1961] Every pair of effectively inseparable sets is m-complete.

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On the other hand,

Theorem

There exists an e.i. ceer that is not universal.

The proof that the u.e.i. ceers are universal relies on:

Theorem

For a given ceer R, the following are equivalent:

- *R is u.e.i.;*
- *R is weakly ufp (meaning: there exists a total computable function $f(D, e, x)$ such that for every finite set D where $[i]_R \neq [j]_R$ for every $i, j \in D$, and every e, x,*

$$\varphi_e(x)\downarrow \in [D]_R \Rightarrow \varphi_e(x) \; R \; f(D, e, x).)$$

- *R is strongly u.m.c. (meaning: for every ceer S and for every pair of numbers a_0, a_1, we have that every partial monomorphism $\pi : S \longrightarrow R$ defined on $\{[a_0]_S, [a_1]_S\}$ can be extended uniformly to a total monomorphism μ, provided that $[a_0]_S \neq [a_1]_S$.) (Clearly strongly u.m.c. ceers are universal.)*

The proof makes again use of infinite sequences of indices that we control by the Recursion Theorem.

Looking for characterizations of universal ceers

Problem

Is it true that the u.e.i. coincide with the ufp ceers?

An affirmative answer would be nice, since it would tie again effective inseparabillity for ceers with provable equivalence in Peano Arithmetic, since the ufp ceers are (up to isomorphisms) the c.e. extensions of PA.

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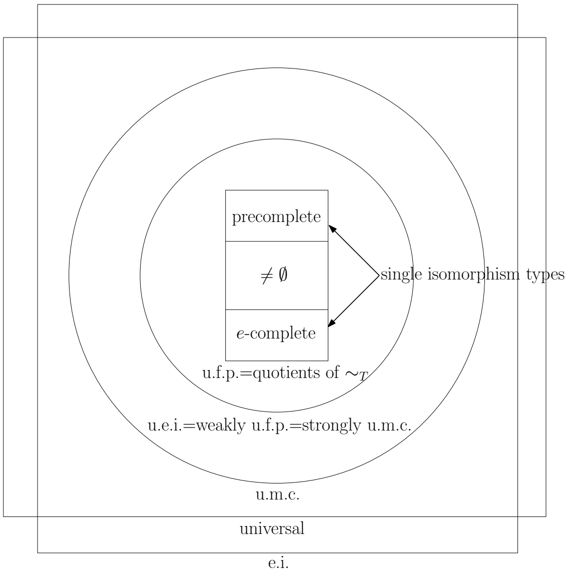
Characterizing universal ceers:

Theorem

A ceer R is universal iff R has a u.e.i. subobject $S \leq R$.

Of course, if R is a universal ceer, then for every ceer S, we have that $R \oplus S$ is also universal. So there are universal ceers that are not u.e.i.

Summary of universal ceers



Index sets of universal ceers

Let

- Univ $= \{e : R_e \text{ universal}\}$;
- uei $= \{e : R_e \text{ u.e.i.}\}$.

Theorem

Univ is Σ^0_3-complete.

Theorem

uei *is* Σ^0_3*-complete.*

Problem

Show that ei is Π^0_4-complete, where ei is the index set of all ceers that yield a partition in e.i. sets.