Enumeration Degree Spectra

The Incomputable
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The enumeration reducibility

Definition. The set A is *enumeration reducible to* the set B ($A \leq_e B$), if $A = W_e(B)$ for some enumeration operator W_e.

Definition. Given a set A, denote by $A^+ = A \oplus (\omega \setminus A)$.

Theorem. *For any sets A and B:*

1. *A is c.e. in B iff $A \leq_e B^+$.*
2. *$A \leq_T B$ iff $A^+ \leq_e B^+$.*

Definition. A set A is called *total* iff $A \equiv_e A^+$.

The standard embedding: $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$.

Degree spectra

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ *be a countable structure. An enumeration of* $\mathfrak{A}$ *is every surjective mapping of* ω *onto* A*.*

Given an enumeration f *of* $\mathfrak{A}$ *and a subset of* B *of* A^a*, let*

$$f^{-1}(B) = \{\langle x_1, \ldots, x_a \rangle \mid (f(x_1), \ldots, f(x_a)) \in B\}.$$

$$f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq).$$

Definition.[Richter] *The Turing degree spectrum of* $\mathfrak{A}$ is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) \mid f \text{ is a one-to-one enumeration of } \mathfrak{A}\}.$$

The least element of $DS_T(\mathfrak{A})$ is called the *degree of* $\mathfrak{A}$.

Enumeration degree spectra

Definition.[Soskov] *The enumeration degree spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

The least element of $DS(\mathfrak{A})$ is called the *e-degree of* $\mathfrak{A}$.

Proposition. *The enumeration degree spectrum is closed upwards with respect to total e-degrees, i.e. if* $\mathbf{b}$ *is a total e-degree and* $\mathbf{a} \leq_e \mathbf{b}$ *for some* $\mathbf{a} \in DS(\mathfrak{A})$*, then* $\mathbf{b} \in DS(\mathfrak{A})$.

Total structures

Definition. The structure $\mathfrak{A}$ is called *total* if for every total enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total.

Proposition. *If $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A}) = \iota(DS_T(\mathfrak{A}))$.*

Given a structure $\mathfrak{A} = (A, R_1, \dots, R_k)$, for every j denote by R_j^c the complement of R_j and let $\mathfrak{A}^+ = (A, R_1, \dots, R_k, R_1^c, \dots, R_k^c)$.

Proposition.

- $\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.
- *If $\mathfrak{A}$ is total then $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$.*

Co-spectra

Definition. Let $\mathcal{A}$ be a nonempty set of enumeration degrees. The *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

Definition. The co-spectrum of the structure $\mathfrak{A}$ is the set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$.
The greatest element of $CS(\mathfrak{A})$ we call the *co-degree* of $\mathfrak{A}$.

Every degree of $\mathfrak{A}$ is a co-degree of $\mathfrak{A}$ as well. The vice versa is not always true.

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Definition. The nth jump spectrum of $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})^{(n)}) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

The least element of $DS_n(\mathfrak{A})$ is called the *nth jump degree of* $\mathfrak{A}$.

Definition. The co-set $CS_n(\mathfrak{A})$ of the nth jump spectrum of $\mathfrak{A}$ is called nth jump co-spectrum of $\mathfrak{A}$.
The greatest element of $CS_n(\mathfrak{A})$ is called the *nth jump co-degree of* $\mathfrak{A}$.

Some examples

Example. *[Richter] Let $\mathfrak{A} = (A; <)$ be a linear ordering. $DS(\mathfrak{A})$ contains a minimal pair of degrees and hence $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$. $\mathbf{0}_e$ is the co-degree of $\mathfrak{A}$. So, if $\mathfrak{A}$ has a degree $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.*

Example. *[Knight] For a linear ordering $\mathfrak{A}$, $CS_1(\mathfrak{A})$ consists of all e-degrees of Σ^0_2 sets. The first jump co-degree of $\mathfrak{A}$ is $\mathbf{0}'_e$.*

Example. *[Slaman,Whener] There exists a structure $\mathfrak{A}$ s.t.*

$$DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ has co-degree $\mathbf{0}_e$ but has not a degree.

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Clearly the structure $\mathfrak{A}$ *has co-degree* $\mathbf{0}_e$ *but has not a degree.*

Representing the principle countable ideals as co-spectra

Example. *[Coles, Downey, Slaman] Let G be a torsion free abelian group of rank 1, i.e. G is a subgroup of Q.*
There exists an enumeration degree $\mathbf{s}_G$ such that

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is $\mathbf{s}_G$.*
- *G has a degree iff $\mathbf{s}_G$ is a total e-degree.*
- *If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the n-th jump degree of G.*

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$.

Corollary. *Every principle ideal of enumeration degrees is $CS(G)$ for some G.*

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Representing non-principle countable ideals as co-spectra

Example. *Let $B_0, \dots, B_n, \dots$ be a sequence of sets of natural numbers. Set $\mathfrak{A} = (\mathbb{N}; f; \sigma)$,*

$$f(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \ \&\ i \in B_k\}.$$

Then $CS(\mathfrak{A}) = I(d_e(B_0), \dots, d_e(B_n), \dots)$

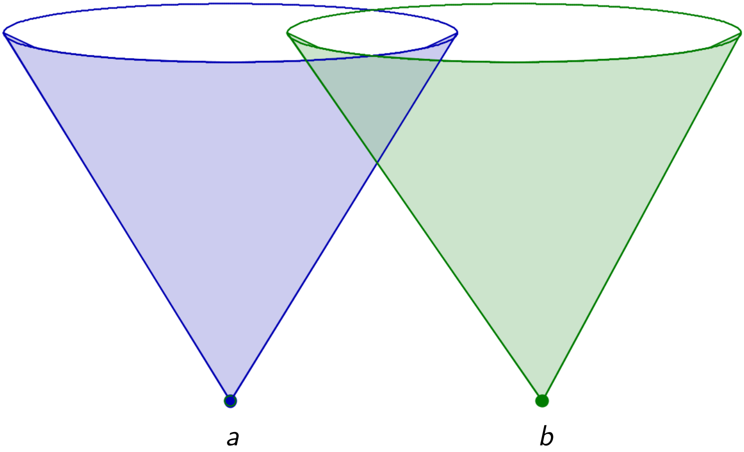
Spectra with a countable base

Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem. *A structure $\mathfrak{A}$ has an e-degree if and only if $DS(\mathfrak{A})$ has a countable base.*

An upwards closed set of degrees which is not a degree spectra of a structure



Properties of upwards closed sets

Theorem. *[Selman]* $\mathbf{a} \leq_e \mathbf{b}$ *iff for all total* $\mathbf{c}$ $(\mathbf{b} \leq_e \mathbf{c} \Rightarrow \mathbf{a} \leq_e \mathbf{c})$.

Proposition. *Let* $\mathcal{A} \subseteq \mathcal{D}_e$ *be a upwards closed set with respect to total e-degrees. Denote by* $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A}\ \&\ \mathbf{a} \text{ is total}\}$. *Then* $co(\mathcal{A}) = co(\mathcal{A}_t)$.

Specific properties of the degree spectra

Theorem. *Let $\mathfrak{A}$ be a structure, $1 \leq n$ and $\mathbf{c} \in DS_n(\mathfrak{A})$. Then*

$$CS(\mathfrak{A}) = co(\{\mathbf{b} \in DS(\mathfrak{A}) : \mathbf{b}^{(n)} = \mathbf{c}\}).$$

Example. *(Upwards closed set for which the Theorem is not true)*

Let $B \not\leq_e A$ and $A \not\leq_e B'$. Let

$$\mathcal{D} = \{\mathbf{a} : d_e(A) \leq_e \mathbf{a}\} \cup \{\mathbf{a} : d_e(B) \leq_e \mathbf{a}\}.$$

Set $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \&\ \mathbf{a}' = d_e(B)'\}$.

- *$d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$.*
- *$d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.*

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The minimal pair theorem

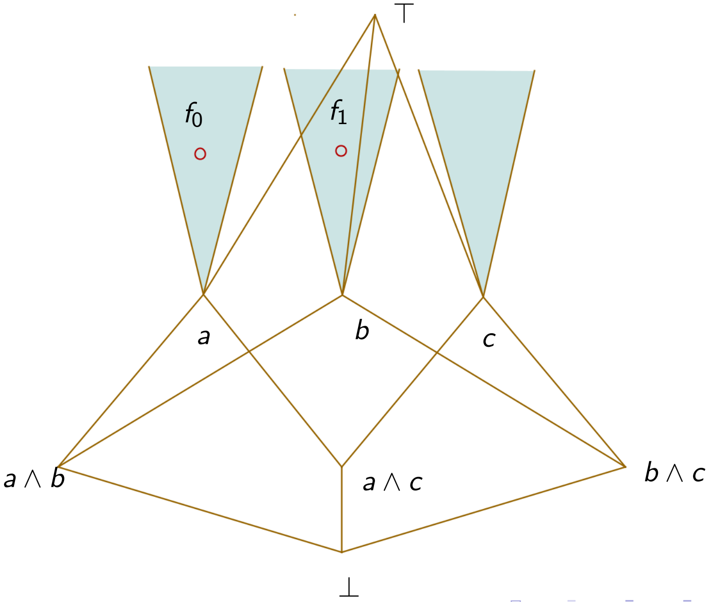
Theorem. *Let* $\mathbf{c} \in DS_2(\mathfrak{A})$. *There exist* $\mathbf{f},\mathbf{g} \in DS(\mathfrak{A})$ *s.t.* $\mathbf{f},\mathbf{g}$ *are total,* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ *and* $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$.

Notice that for every enumeration degree $\mathbf{b}$ *there exists a structure* $\mathfrak{A}_{\mathbf{b}}$ *s. t.* $DS(\mathfrak{A}_{\mathbf{b}}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{b} <_e \mathbf{x}\}$. *Hence*

Corollary. *[Rozinas] For every* $\mathbf{b} \in \mathcal{D}_e$ *there exist total* $\mathbf{f}, \mathbf{g}$ *below* $\mathbf{b}''$ *which are a minimal pair over* $\mathbf{b}$.

Not every upwards closed set of enumeration degrees has a minimal pair:

An upwards closed set with no minimal pair



The quasi-minimal degree

Definition. Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

From Selman's theorem it follows that if $\mathbf{q}$ *is quasi-minimal with respect to* $\mathcal{A}$*, then* $\mathbf{q}$ *is an upper bound of* $co(\mathcal{A})$*.*

The quasi-minimal degree for a degree spectra

Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Corollary. *[Slaman and Sorbi] Let I be a countable ideal of enumeration degrees. There exists an enumeration degree $\mathbf{q}$ s.t.*

1. *If $\mathbf{a} \in I$ then $\mathbf{a} <_e \mathbf{q}$.*
2. *If $\mathbf{a}$ is total and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in I$.*

Properties of the quasi-minimal degrees

Proposition. *For every countable structure $\mathfrak{A}$ there exist continuum many quasi-minimal degrees with respect to $DS(\mathfrak{A})$.*

Proposition. *The first jump spectrum $DS_1(\mathfrak{A})$ of every structure $\mathfrak{A}$ consists exactly of the enumeration jumps of the quasi-minimal degrees with respect to $DS(\mathfrak{A})$.*

Corollary. *[McEvoy] For every total e-degree $\mathbf{a} \geq_e \mathbf{0}'_e$ there is a quasi-minimal degree $\mathbf{q}$ with $\mathbf{q}' = \mathbf{a}$.*

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Splitting a total set

Proposition. *[Jockusch] For every total e-degree* $\mathbf{a}$ *there are quasi-minimal degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Proposition. *For every element* $\mathbf{a}$ *of the jump spectrum of a structure* $\mathfrak{A}$ *there exists quasi-minimal with respect to* $DS(\mathfrak{A})$ *degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Every jump spectrum is the spectrum of a total structure

Let $\mathfrak{A} = (A; R_1, \ldots, R_n)$.
Let $\bar{0} \notin A$. *Set* $A_0 = A \cup \{\bar{0}\}$. *Let* $\langle ., . \rangle$ *be a pairing function s.t. none of the elements of* A_0 *is a pair and* A^* *be the least set containing* A_0 *and closed under* $\langle ., . \rangle$.

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (A^*, R_1, \ldots, R_n, A_0, G_{\langle .,. \rangle}).$$

Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)[x \in W_e(\tau^{-1}(\mathfrak{A}))]\}$.
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, A^* \setminus K_{\mathfrak{A}})$.

Theorem.

1. *The structure* $\mathfrak{A}'$ *is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

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Theorem.

1. *The structure* $\mathfrak{A}'$ *is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$*.*

The jump inversion theorem

Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

$$DS(\mathfrak{B})_t = \{\mathbf{a} | \mathbf{a} \in DS(\mathfrak{B}) \text{ and } \mathbf{a} \text{ is total}\} \subseteq DS_1(\mathfrak{A}).$$

Theorem. *There exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{B})_t$.*

Method: Marker's extensions.

Corollary. *Let $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. Then there exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}'$. Then there exists a total structure $\mathfrak{C}$ such that $DS(\mathfrak{B}) = DS(\mathfrak{C})$.*

Remark.

2009 *Montalban, Notes on the jump of a structure, Mathematical Theory and Computational Practice, 372–378.*

2009 *Stukachev, A jump inversion theorem for the semilattices of Sigma-degrees, Siberian Electronic Mathematical Reports, v. 6, 182 – 190*

2012 *Montalban, Rice Sequences of Relations, to appear in the Philosophical Transactions A.*

Applications

Example. *[Ash, Jockusch, Knight and Downey] Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ s.t. $\mathfrak{C}$ has a $n+1$-th jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.*
It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

1. *$DS(\mathfrak{B})$ has not a least element.*
2. *$\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.*
3. *All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.*

Consider a set B satisfying:

1. *B is quasi-minimal above $\mathbf{0}^{(n)}$.*
2. *$B' \equiv_e \mathbf{0}^{(n+1)}$.*

Let G be a subgroup of the additive group of the rationals s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} \mid d_e(S_G) \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.

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Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = \{\mathbf{a} \mid \mathbf{0}^{(n)} <_e \mathbf{a}\}$.
It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total e-degrees greater than $\mathbf{0}^{(n)}$.
This could be done by Whener's construction using a special family of sets:

Theorem. *Let $n \geq 0$. There exists a family $\mathcal{F}$ of sets of natural numbers s.t. for every X strictly above $\mathbf{0}^{(n)}$ there exists a computable in X set U satisfying the equivalence:*

$$F \in \mathcal{F} \iff (\exists a)(F = \{x | (a, x) \in U\}).$$

But there is no such U c.e. in $\mathbf{0}^{(n)}$.

Enumeration degree spectra

- Questions:
  - Describe the sets of Turing degrees (enumeration degrees) which are equal to $DS(\mathfrak{A})$ for some structure $\mathfrak{A}$.
  - Is the set of all Muchnik degrees containing some degree spectra definable in the lattice of the Muchnik degrees?

Thank you!