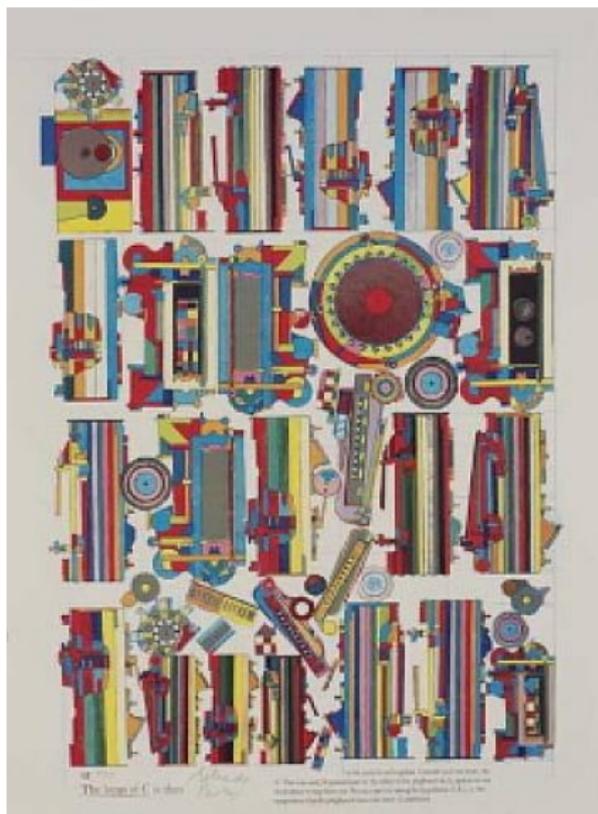# Generalised transfinite Turing machines and strategies for games.

P.D.Welch, University of Bristol.   Chicheley Hall 2012

• Theme: Connections between inductive operators, discrete transfinite machine models of computation, and determinacy.

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- Part I: ITTM description.
- Part II: Fixed points of operators yielding some strategies.
- Part III: Generalising Operators and Machines

Part I: ITTM description[1]

- Allow a standard Turing machine to run transfinitely using one of the usual programs $\langle P_e \mid e \in \mathbb{N} \rangle$.
- Alphabet: $\{0, 1\}$;
- Enumerate the cells of the tape $\langle C_k \mid k \in \mathbb{N} \rangle$.

    Let the current instruction about to be performed at time τ be $I_{i(\tau)}$;
    Let the current cell being inspected be $C_{p(\tau)}$.

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- Behaviour at successor stages $\alpha \to \alpha + 1$: as normal.

At limit times λ: (a) we specify cell values by:

$$C_k(\lambda) = Liminf_{\beta \to \lambda} C_k(\alpha)$$

(where the value in C_k at time τ is $C_k(\tau)$).

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(b) we also (i) put the R/W to cell $C_{p(\lambda)}$ where

$$p(\lambda) = Liminf^*_{\alpha \mapsto \lambda} \{ p(\beta) \mid \alpha < \beta < \lambda \};$$

(ii) set

$$i(\lambda) = Liminf_{\alpha \mapsto \lambda} \{ i(\beta) \mid \alpha < \beta < \lambda \}.$$

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• Hamkins & Lewis proved there is a *universal machine*, an S_n^m-*Theorem*, and a *Recursion Theorem* for ITTM's, and a wealth of results on the resulting ITTM-*degree theory*.
• We may define halting sets:

$$H = \{(e, x) \mid e \in \mathbb{N}, x \in 2^{\mathbb{N}} \wedge P_e(x) \downarrow\}$$

$$H_0 = \{(e, 0) \mid e \in \mathbb{N} \wedge P_e(0) \downarrow\}$$

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Q. What is H or H_0?

Q. How long do we have to wait to discover if $e \in H_0$ or not?

Q. What are the ITTM (semi)-decidable sets of integers? Or reals?

A Kleene Normal Form Theorem?

We'd like some type of a *"ITTM Normal Form Theorem"*:

Theorem
There is a universal predicate $\mathfrak{T}$ *which satisfies* $\forall e \forall x$:

$$P_e(x) \downarrow z \quad \leftrightarrow \quad \exists y \in 2^{\mathbb{N}} [\mathfrak{T}(e, x, y) \wedge Last(y) = z].$$

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• However for this to occur we need to know whether the ordinal *length* of any computation is capable of being output or *written* by a(nother) computation.

The λ, ζ, Σ-Theorem[2]

Theorem
Let ζ be the least ordinal so that there exists $\Sigma > \zeta$ with the property that

$$L_\zeta \prec_{\Sigma_2} L_\Sigma; \qquad (\zeta \text{ is "}\Sigma_2\text{-extendible".})$$

(i) Then the universal ITTM *on integer input first enters a loop at time ζ.*

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Let λ be the least ordinal satisfying:

$$L_\lambda \prec_{\Sigma_1} L_\zeta.$$

(ii) Then $\lambda = sup\{\alpha \mid \exists e \ P_e(0){\downarrow} \text{ in } \alpha \text{ steps}\}$
$\qquad = sup\{\alpha \mid \exists e \ P_e(0){\downarrow} y \in WO \wedge ||y|| = \alpha\}.$

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• As a corollary one derives the Normal Form Theorem and:

Corollary

$$H_0 \equiv \Sigma_1\text{-}Th(L_\lambda).$$

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Part II: Over $\mathbb{N}$ $\Pi_1^1\text{-}IND = \partial\Sigma_1^0$

- The *game* quantifier ∂:

Definition

A set $A \subseteq \mathbb{N}$ is $\partial\Gamma$ if there is $B \in \mathbb{N} \times \mathbb{R}$ so that:

$$n \in A \iff \text{Player } I \text{ has a winning strategy in } G_{B_n}$$

where $B_n = \{x \in \mathbb{R} \mid B(n,x)\}$.

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A set $A \subseteq \mathbb{N}$ is (mon.-Π_1^1)-IND iff it is $\eth\Sigma_1^0$ iff it is $\Sigma_1(L_{\omega_1^{ck}})$.

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Theorem (Svenonius)
Any Σ_1^0 game, if it is a win for Player I then she has a (mon.-Π_1^1)-IND winning strategy. Hence this is also $\Sigma_1(L_{\omega_1^{ck}})$.

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Q Are strategies for Σ_3^0 sets ITTM-semi-decidable? Thus: are they $\Sigma_1(L_\Sigma)$?

Definition
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Theorem (W)
The theories:

$$\Pi_3^1\text{-}\mathsf{CA}_0, \Delta_3^1\text{-}\mathsf{CA}_0 + \Sigma_3^0\text{-}\mathsf{Det}, \Delta_3^1\text{-}\mathsf{CA}_0 + \mathsf{ITTM}, \Delta_3^1\text{-}\mathsf{CA}_0$$

are in strictly descending order of strength[3].

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Open Questions

Q1 *Give another description of the least β over which strategies for Σ_3^0 sets are definable.*

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Conjecture 1: Lubarsky's 'Feedback-ITTM's' are related to this.

Lubarsky's Feedback ITTM's[4]

Lubarsky has suggested a variant of the HL-machine which is allowed to make calls to sub-routines to obtain the answer to the question:

"*Does the ITTM-computation P_e with the current real x of the scratch tape halt or loop?*".

• In other words one considers ITTM-computations recursive in *H*.

• Clearly such an FITTM may make an infinite chain of such calls in which case Lubarsky calls the computation "freezing".

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• Clearly such an FITTM may make an infinite chain of such calls in which case Lubarsky calls the computation "freezing".

• The natural *Conjecture 1* emerges that any winning strategy for a Σ^0_3 game which wins for player I can be written by an FITTM.

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- Does $\beta =$ the closure ordinal of monotone-$\eth\Pi_3^0$-Inductive Operators?

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(i) If $\gamma < \beta$ is least with $L_\gamma \prec_{\Sigma_1} L_\beta$ then γ is the closure ordinal of monotone-$\eth\Sigma_3^0$-Inductive Operators.
(ii) Σ_1-$Th(L_\gamma)$ is a complete $\eth\Sigma_3^0$ set.

Part III: Hypermachines[5]

• Can we find 'machines' that will lift the Σ_2 "*Liminf*" of [HL] to a Σ_n-rule at limit stages?

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For any $n \geq 2$ there is such a Σ_n limit rule, so that for a machine running under such a rule, the universal Σ_n-machine on integer input first enters a loop at the least $\zeta(n)$ s.t. $\exists\Sigma(n) > \zeta(n)$ with

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• Such machines then compute, taken as a whole, all the reals of the least β-model of analysis $2^\omega \cap L_{\beta_0}$.

• Then, *e.g.* using Montalban-Shore, strategies for n-Σ_3^0 games are computable by the Σ_{n+2}-machines.

Open Questions continued

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Q4 *Quantify Det($\mathbf{\Gamma_n}$).*

A sample theorem of what is known:

Theorem
ZFC + Det($\mathbf{\Gamma_2}$) $\Rightarrow$ There is an inner model with a proper class of strong cardinals[6].

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[6]W: *Determinacy in Strong Cardinal Models*, JSL, June 2011