Cayley graph automatic groups are not necessarily Cayley graph biautomatic

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$\mathbb{Z} \times \mathbb{Z}$ is automatic

$$a = (1,0), \qquad b = (0,1), \qquad S = \{a, a^{-1}, b, b^{-1}\}$$

We need a finite automaton recognazing a language of normal forms in our group

$$L = \{a^m b^n \mid m, n \in \mathbb{Z}\} \subseteq S*$$

We need 4 automata, one for each of the 4 generators a, a^{-1}, b, b^{-1}, recognizing the language of multiplication (on the right) by the corresponding generator.

$$L_a = \binom{a}{a}^* \binom{b}{a} \binom{b}{b}^* \binom{\diamond}{b} \cup \binom{a^-1}{a^-1}^* \binom{a^{-1}}{b} \binom{b}{b}^* \binom{b}{\diamond} \cup \cdots \subseteq ((S_\diamond)^2)^*$$

Automatic groups

- Based on ideas of Thurston, Cannon, Gilman, Epstein, Holt, ...
"Word Processing in Groups" in 1992.

- Motivation: provide a framework for calculations in fundamental groups of compact 3-manifolds.

- Excellent algorithmic properties (quadratic Dehn function)

- Excellent geometric properties (fellow traveling of geodesics)

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- Automatic groups cannot handle manifolds of Nil and Sol type

- A nilpotent group is automatic if and only if it is virtually abelian.

Automatic groups

- Do all automatic groups have decidable Conjugacy Problem?

- Is the Isomorphism Problem decidable in the class of automatic groups?

- Are all automatic groups biautomatic?

Generalization(s)

- Bridson, Gilman generalization (1996) covers all compact 3-manifolds but the good algorithmic properties are lost.

- Kharlampovich, Khoussainov, Miasnikov (2011) keep good algorithmic properties, but the geometry is lost.

Cayley automatic groups

- Word Problem decidable in quadratic time.

- Conjugacy Problem decidable in the biautomatic case.

- Many nilpotent (Heisenberg,...) and solvable ($BS(1, n)$, ...) groups are Cayley automatic.

- Good closure properties: direct products, free products, some amalgamations (e.g., over finite groups), finite extensions, some semidirect products (if the action is automatic).

- Many proofs become simpler.

The main results

- There are Cayley automatic groups with undecidable Conjugacy Problem.

- There are Cayley automatic groups that are not Cayley biautomatic.

- The Isomorphism Problem is not decidable in the class of Cayley automatic groups.

Key ingredients

Theorem (Bogopolski-Martino-Ventura, 2010)

There exists a group of the form $\mathbb{Z}^d \rtimes_\tau F_n$ with undecidable Conjugacy Problem.

The "secret" is that $\tau(F_n) \leq GL_d(\mathbb{Z})$ has undecidable Orbit Problem (it is not decidable if, given vectors u and v in $\mathbb{Z}^d$, there is a matrix A in $\tau(F_n)$ such that $u^A = v$).

Theorem (Ventura-Š, 2010, unpublished)

There exists a group of the form $\mathbb{Z}^d \rtimes_\tau F_n$ with undecidable Conjugacy Problem such that τ is injective.

Key ingredients

Theorem (Levitt, 2008 unpublished)

The Isomorphism Problem is not decidable in the class of groups of the form $\mathbb{Z}^d \rtimes F_n$.

Theorem (Miasnikov-Š, 2012)

If a group of the form $\mathbb{Z}^d \rtimes F_n$ *has subexponential Dehn function, then it has decidable Conjugacy Problem.*

(Thus, our examples are not automatic in the classical sense.)

Convolution of words

For Σ a finite alphabet, denote $\Sigma_\diamond = \Sigma \cup \{\diamond\}$.
For an n-tuple of words $(w_1, \dots, w_n)$ over Σ define the *convolution* $\otimes(w_1, \dots, w_n)$ to be the word of length $\max\{|w_1|, \dots, |w_n|\}$ over $(\Sigma_\diamond)^n$ in which the j-th symbol is $(\sigma_1, \dots, \sigma_n)$, where

$$\sigma_i = \begin{cases} \text{the } j\text{-th symbol of } w_i, & \text{if } j \leq |w_i| \\ \diamond, & \text{otherwise} \end{cases}.$$

For instance,

$$\otimes(aaa, babaa, \emptyset) = \begin{pmatrix} a \\ b \\ \diamond \end{pmatrix} \begin{pmatrix} a \\ a \\ \diamond \end{pmatrix} \begin{pmatrix} a \\ b \\ \diamond \end{pmatrix} \begin{pmatrix} \diamond \\ a \\ \diamond \end{pmatrix} \begin{pmatrix} \diamond \\ a \\ \diamond \end{pmatrix},$$

where $\emptyset$ denotes the empty word and the symbols in $(\Sigma_\diamond)^n$ are written, for convenience, as columns.

Convolution of relations

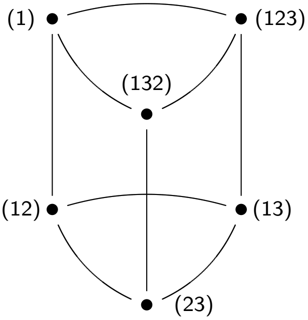
Let R be an n-ary relation on Σ^*. The *convolution* $\otimes R$ of R is the language over $(\Sigma_\diamond)^n$ defined by

$$\otimes R = \{ \otimes(w_1, \dots, w_n) \mid (w_1, \dots, w_n) \in R \}.$$

A relation R is *regular* over Σ if its convolution $\otimes R$ is a regular language over $(\Sigma_\diamond)^n$, i.e., $\otimes R$ is recognizable by a finite automaton over the alphabet $(\Sigma_\diamond)^n$.

Cayley graphs

The (right) Cayley graph of the symmetric group Sym_3 with respect to the generating set $S = \{a, a^{-1}, b\}$, where $a = (123)$ and $b = (12)$, is given by



Cayley graphs (as systems of relations)

Let G be a finitely generated group with finite generating set S. The right Cayley graph of G with respect to S is the graph $\Gamma(G, S)$ with G as the set of vertices and, for each g in G and s in S, an edge from g to gs. The Cayley graph can be interpreted as a system of $|S|$ binary relations E_s on G, for s in S, where

$$E_s = \{ \ (g, gs) \mid g \in G \ \}.$$

A map $\bar{\ } : G \to \Sigma^*$ induces $|S|$ binary relations on Σ^* given by

$$\overline{E}_s = \{ \ (\overline{g}, \overline{gs}) \mid g \in G \ \}.$$

Definition of a Cayley automatic group

Definition

A finitely generated group G with finite generating set S is *Cayley automatic* if there exists a finite alphabet Σ and an injective map $\bar{\ }: G \to \Sigma^*$ such that
$\overline{G}$ is regular (over Σ) and
$\overline{E}_s$ is regular (over Σ), for every s in S.
In such a case the tuple $(\overline{G}, \overline{E}_{s_1}, \ldots, \overline{E}_{s_k})$ is called an *automatic structure* of the Cayley graph $\Gamma(G, S)$ or Cayley automatic structure of G (with respect to $S = \{s_1, \ldots, s_k\}$).

$\mathbb{Z}$ is Cayley automatic

The elements (i.e., their "normal forms"):

$$+\sigma\sigma\dots\sigma 1 \qquad -\sigma\sigma\dots\sigma 1 \qquad +0$$

Addition of 1:

$$\begin{array}{l} +11\cdots 1\,0\,\sigma\sigma\cdots\sigma\,1 \\ +\underbrace{00\cdots 0}_{*}\,1\,\underbrace{\sigma\sigma\cdots\sigma\,1}_{*} \end{array} \qquad \begin{array}{l} +11\cdots 1\,\diamond \\ +\underbrace{00\cdots 0\,1}_{+} \end{array} \qquad \begin{array}{l} +0 \\ +1 \end{array}$$

$$\begin{array}{l} -00\cdots 0\,1\,\sigma\sigma\cdots\sigma\,1 \\ -\underbrace{11\cdots 1}_{*}\,0\,\underbrace{\sigma\sigma\cdots\sigma\,1}_{*} \end{array} \qquad \begin{array}{l} -00\cdots 0\,1 \\ -\underbrace{11\cdots 1\,\diamond}_{+} \end{array} \qquad \begin{array}{l} -1 \\ +0 \end{array}$$

Cayley biautomatic groups

The left Cayley graph can be interpreted as a system of $|S|$ binary relations E^ℓ_s on G, for s in S, where

$$E^\ell_s = \{ (g, sg) \mid g \in G \}.$$

Definition

A finitely generated group G with finite generating set S is *Cayley biautomatic* if there exists a finite alphabet Σ and an injective map $\overline{} : G \to \Sigma^*$ such that
$\overline{G}$ is regular (over Σ),
$\overline{E}_s$ is regular (over Σ), for every s in S, and
$\overline{E}^\ell_s$ is regular (over Σ), for every s in S.
In such a case the tuple $(\overline{G}, \overline{E}_{s_1}, \dots, \overline{E}_{s_k}, \overline{E}^\ell_{s_1}, \dots, \overline{E}^\ell_{s_k})$ is called a *biautomatic structure* of the pair of Cayley graphs $\Gamma(G, S)$ and $\Gamma^\ell(G, S)$ or Cayley biautomatic structure of G (with respect to $S = \{s_1, \dots, s_k\}$).

French poodles, matrix multiplication, automatic actions

$\mathbb{Z}^d \rtimes F_n$ is Cayley automatic because we can teach a French poodle to multiply by (finitely many) integer matrices.

$$(M, u)v = (M, u + v)$$
$$(M, u)S_i = (MS_i, u^{S_i})$$

But, is not Cayley biautomatic since we cannot teach the poodle to multuply by all integer matrices.

$$S_i(M, u) = (S_i M, u)$$
$$v(M, u) = (M, v^M + u)$$

The main results

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- There are Cayley automatic groups that are not Cayley biautomatic.

- The Isomorphism Problem is not decidable in the class of Cayley automatic groups.