k-automatic sets of rational numbers

Eric Rowland[1]   Jeffrey Shallit[2]

[1]LaCIM, Université du Québec à Montréal

[2]University of Waterloo

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Notation

Let $k \geq 2$ be an integer.

- $\Sigma_k = \{0, 1, \ldots, k-1\}$
- $\mathbb{N} = \{0, 1, 2, \ldots\}$

Given a word $w = a_1 a_2 \cdots a_t \in \Sigma_k^*$, let

$$[w]_k = \sum_{1 \leq i \leq t} a_i k^{t-i}.$$

For example, $[101011]_2 = 43$.

Also, $[0101011]_2 = 43$.

Automatic sets of integers

Given a language $L \subseteq \Sigma_k^*$, define

$$[L]_k = \{[w]_k \,:\, w \in L\}$$

to be the set of integers it represents.

Definition

A set $S \subseteq \mathbb{N}$ is *k-automatic* if there exists a regular language $L \subseteq \Sigma_k^*$ such that $S = [L]_k$.

Representing rationals

We allow the rational number p/q to be represented by *any* pair of integers (p', q') with $p/q = p'/q'$.

We represent (p, q) as a word $w = [a_1, b_1][a_2, b_2] \cdots [a_n, b_n]$ over Σ_k^2.

Define projection maps π_1, π_2 as follows:

$$\pi_1(w) = a_1 a_2 \cdots a_n; \qquad \pi_2(w) = b_1 b_2 \cdots b_n.$$

Given a word $w \in (\Sigma_k^2)^*$ with $[\pi_2(w)]_k \neq 0$, define

$$\mathrm{quo}_k(w) := \frac{[\pi_1(w)]_k}{[\pi_2(w)]_k}.$$

Example

If $w = [1, 0][0, 1][1, 0][0, 0][1, 1][1, 0]$, then $\mathrm{quo}_2(w) = 43/18$.
Also, $\mathrm{quo}_2([0, 0]w) = 43/18$.
Also, $\mathrm{quo}_2(w[0, 0]) = 86/36 = 43/18$.

Automatic sets of rationals

Given a language $L \subseteq (\Sigma_k^2)^*$ such that $[\pi_2(w)]_k \neq 0$ for all $w \in L$, define

$$\mathrm{quo}_k(L) := \{\mathrm{quo}_k(w) \ : \ w \in L\}$$

to be the set of rationals it represents.

Definition

A set $S \subseteq \mathbb{Q}^{\geq 0}$ is *k-automatic* if there exists a regular language $L \subseteq (\Sigma_k^2)^*$ such that $S = \mathrm{quo}_k(L)$.

We write that S is $(\mathbb{N}, k)$-automatic or $(\mathbb{Q}, k)$-automatic when it is necessary to distinguish the two notations of automaticity.

Examples

Example

Let $k = 2$, and let $A = \{[0,0],[0,1],[1,0],[1,1]\}$.
Consider L defined by the regular expression $A^*\{[0,1],[1,1]\}A^*$.
Then $\mathrm{quo}_k(L) = \mathbb{Q}^{\geq 0}$.

Example

Consider the regular language

$$L = \{w \in (\Sigma_k^2)^* : \pi_1(w) \in \Sigma_k^* \text{ and } [\pi_2(w)]_k = 1\}.$$

Then $\mathrm{quo}_k(L) = \mathbb{N}$.

Examples

Example

Let L be defined by the regular expression $[0,1]\{[0,0],[2,0]\}^*$.

Then $\mathrm{quo}_3(L)$ is the 3*-adic Cantor set*, the set of all rational numbers in the "middle-thirds" Cantor set whose denominator is a power of 3.

Example

Loxton and van der Poorten (1987) were interested in the set $T = \{0, 1, 3, 4, 5, 11, 12, 13, \dots\}$ of all non-negative integers that can be represented using only the digits $0, 1, -1$ in base 4.

The set $S = \{p/q \,:\, p, q \in T\}$ is 4-automatic.
They showed that S contains every odd positive integer.

A basic result

Theorem

Let β be a non-negative real number and define

$$L_{\leq\beta} = \{w \in (\Sigma_k^2)^* \ : \ \mathrm{quo}_k(w) \leq \beta\},$$

and analogously for the other relations $<, =, \geq, >, \neq$. Then $L_{\leq\beta}$ (resp., $L_{<\beta}$, $L_{=\beta}$, $L_{\geq\beta}$, $L_{>\beta}$) is regular if and only if β is a rational number.

Closure properties

Theorem

Let $\alpha \in \mathbb{Q}^{\geq 0}$. The class of k-automatic sets of rational numbers is closed under the following operations:
(i) union;
(ii) $S \to S + \alpha := \{x + \alpha \,:\, x \in S\}$;
(iii) $S \to S \dot{-} \alpha := \{\max(x - \alpha, 0) \,:\, x \in S\}$;
(iv) $S \to \alpha \dot{-} S := \{\max(\alpha - x, 0) \,:\, x \in S\}$;
(v) $S \to \alpha S := \{\alpha x \,:\, x \in S\}$;
(vi) $S \to \{1/x \,:\, x \in S \setminus \{0\}\}$.

Unlike the class of $(\mathbb{N}, k)$-automatic sets, the class of $(\mathbb{Q}, k)$-automatic sets is not closed under the operations of intersection or complement.

Theorem

Let $S_1 = \{(k^n - 1)/(k^m - 1) \,:\, 1 \leq m < n\}$ and $S_2 = \mathbb{N}$.
Then $S_1 \cap S_2$ is not k-automatic.

Main result

Theorem

Let $S \subseteq \mathbb{N}$.
Then S is $(\mathbb{N}, k)$-automatic if and only if it is $(\mathbb{Q}, k)$-automatic.

The proof uses a result which is interesting in its own right.

We say a set $S \subseteq \mathbb{N}$ is *ultimately periodic* if there exist integers $n_0 \geq 0, p \geq 1$ such that $n \in S \iff n + p \in S$, provided $n \geq n_0$.

Example

$\{0, 1, 5\}$ is ultimately periodic (as is every finite set).
$\{0, 1, 5, 6, 10, 11, 15, 16, 20, 21, \dots\}$ is ultimately periodic.

Finite sets of primes

Let $\operatorname{pd}(n) = \{p \in \mathbb{P} : p \mid n\}$. For example, $\operatorname{pd}(60) = \{2, 3, 5\}$.
Given a subset $D \subset \mathbb{P}$, let $\pi(D) = \{n \geq 1 \ : \ \operatorname{pd}(n) \subseteq D\}$.
Finally, let $\nu_k(n) := \max\{i \ : \ k^i \mid n\}$.

Theorem

Let $D \subseteq \mathbb{P}$ be a finite set of primes, and let $S \subseteq \pi(D)$.
Then S is k-automatic if and only if

1. $F := \{\frac{s}{k^{\nu_k(s)}} \ : \ s \in S\}$ *is finite, and*
2. *for all $f \in F$ the set $U_f = \{i \ : \ k^i f \in S\}$ is ultimately periodic.*

Example

The set $\{1, 8, 21\} \cup \{3 \cdot 2^{5j} : j \in \mathbb{N}\} \cup \{3 \cdot 2^{5j+1} : j \in \mathbb{N}\}$ is 2-automatic.
The set $\{6^j : j \in \mathbb{N}\}$ is not 2-automatic.

Some non-automatic sets

Let $(p, q)_k$ be the representation of (p, q) in $(\Sigma_k^2)^*$ with no leading $[0, 0]$.

Remark

The following languages are not context-free.

- $L_d = \{(p, q)_k \,:\, q \mid p\}$
- $L_r = \{(p, q)_k \,:\, \gcd(p, q) > 1\}$
- $L_g = \{(p, q)_k \,:\, \gcd(p, q) = 1\}$

Since the condition $\gcd(p, q) = 1$ cannot be checked by automata, this is another reason why we don't only accept "reduced" representations.

Solvability results

Theorem

The following problems are recursively solvable.

- *Given a DFA M, a rational number α, and a relation $\lhd$ chosen from $=, \neq, <, \leq, >, \geq$, does there exist $x \in \mathrm{quo}_k(L(M))$ with $x \lhd \alpha$?*
- *Given a DFA M and an integer k, is $\mathrm{quo}_k(L(M))$ infinite?*

The second question is *not* the same as asking if $L(M)$ is infinite, since a number may have infinitely many representations.

Decidability results

Theorem

Given a regular language $L \subseteq (\Sigma_k^2)^*$, the following are decidable.

- $\mathrm{quo}_k(L) \subseteq \mathbb{N}$.
- $\mathrm{quo}_k(L) = \mathbb{N}$.
- $\mathrm{quo}_k(L) \setminus \mathbb{N}$ *is finite.*

Open problem

Is it decidable whether $\mathrm{quo}_k(L) = \mathbb{Q}^{\geq 0}$?