Covering space in the Besicovitch topology

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LATA 2012 – 6 march

Frame 1

- Goal: Better understanding of the structures of *open balls* in the topological space
- In a compact, the more you have points, the more they are close to each-other
  - $\forall \varepsilon, \exists N, \forall \mathcal{S}, |\mathcal{S}| \geq N \Rightarrow \exists x \in \mathcal{S}, \exists y \in \mathcal{S}, \mathbf{d}(x,y) < \varepsilon$
- Problem 1: For a relation between ε and N
  - Estimate $\varepsilon = \max_{\mathcal{S}|\#\mathcal{S}=N} m(\mathcal{S})$, $m(\mathcal{S}) = \min_{x,y\in\mathcal{S}} \mathbf{d}(x,y)$
- Problem 2: What happens on *non-compact* spaces

Playground 2

- $\{0,1\}^\ell$, set of binary words of given length ℓ
  - Compact space endowed with *Hamming distance*
  - $\mathbf{d}_H(x,y) = |\{i | x_i \neq y_i\}|$
- $\{0,1\}^\infty$, set of (uni-)infinite binary words (Cantor space)
  - We call them *configurations*
  - Compact space endowed with the *Cantor distance*
  - Non-compact space endowed with de *Besicovitch distance*
    - *Extension of Hamming distance to infinite word*
  - $\mathbf{d}_B(x,y) = \limsup_{n\to\infty} \frac{\mathbf{d}_H(x_{-n..n}, y_{-n..n})}{n}$
    - Note that $\mathbf{d}_B(x,y) \in [0,1]$

Hamming distance and $\{0,1\}^\ell$ case 3

Proposition 1 *Let $\mathcal{S}$ be a set of N words of $\{0,1\}^\ell$. Then:*

$$\exists x \in \mathcal{S}, \exists y \in \mathcal{S}, \mathbf{d}_H(x,y) \leq \frac{\left\lceil \frac{N}{2} \right\rceil}{2\left\lceil \frac{N}{2} \right\rceil - 1} \ell$$

Proposition 2 (Semakov, Zinoviev, 68) *For all N, there is a set $\mathcal{W}_N$ of N words of $\{0,1\}^\ell$ for some ℓ such that:*

$$\forall x \in \mathcal{W}_N, \forall y \in \mathcal{W}_N, \mathbf{d}_H(x,y) = \frac{\left\lceil \frac{N}{2} \right\rceil}{2\left\lceil \frac{N}{2} \right\rceil - 1} \ell$$

Idea: All possibles "columns" with $\left\lceil \frac{N}{2} \right\rceil - 1$ ones

Besicovitch distance and $\{0,1\}^\infty$ case 4

Proposition 3 *There exists an uncountable set of configurations $\mathcal{Y}$ such that for any x and y in $\mathcal{Y}$, $\mathbf{d}_B(x,y) \equiv 1$*

The configurations of $\mathcal{Y}$ can be chosen quasiperiodic

- *The configuration x is quasiperiodic if for all n there is a w such that all factor of length n of x can be found in any factor of length w.*

Extension to dense sets 5

Corollary 1 *If $\mathcal{R} \subset \{0,1\}^\infty$ is a dense set, then for all $\varepsilon > 0$, there is an infinite set $\mathcal{E}^\varepsilon_\mathcal{R} \subset \mathcal{R}$ such that for any x and y in $\mathcal{E}^\varepsilon_\mathcal{R}$, $d_B(x,y) > 1-\varepsilon$.*

If $\mathcal{R}$ is uncountable, $\mathcal{E}^\varepsilon_\mathcal{R}$ can be chosen uncountable.

Idea: Simple "ε cutting" strategy: $\mathcal{R}$ is dense hence close to members of $\mathcal{Y}$

Why this negative result 6

- Choice of a fast increasing functions so that prefix values are negligible ($x \mapsto 2^{2^x}$)
- Besicovitch distance is defined using prefixes of x without taking the whole configuration into account
- Use of *Toeplitz configurations*

Toeplitz configuration set $\mathcal{T}$ 7

Toeplitz configuration properties 8

- Dense set in Besicovitch topology
  - Can build set $\mathcal{E}^\varepsilon_\mathcal{T}$
- At step n, a proportion of $\sum_{0\leq j\leq n} \frac{1}{p_j}$ cells are fixed
- Not a big restriction on configurations
  - Can take the period p_n of step n be $\mathsf{Ack}(p_{n-1}, p_{n-1})$

Regular Toeplitz configurations 9

- *Regular Toeplitz configurations* are such that the periods are constrained by: $\sum \frac{1}{p_j} = 1$
- During the process, the proportions of assigned cells tends to 1
- No possibility of *"almost only first assigned cells"*

Result for regular Toeplitz configurations 10

Proposition 4 *Let $\mathcal{S}$ be a set of N regular Toeplitz configurations. Then:*

$$\exists x \in \mathcal{S}, \exists y \in \mathcal{S}, \mathbf{d}_B(x,y) \leq \frac{\left\lceil \frac{N}{2} \right\rceil}{2\left\lceil \frac{N}{2} \right\rceil - 1}$$

Proposition 5 *For all integer N, there is a set $\mathcal{X}_N$ of N periodic (hence regular Toeplitz) configurations such that*

$$\forall x \in \mathcal{X}_N, \forall y \in \mathcal{X}_N, \mathbf{d}_B(x,y) = \frac{\left\lceil \frac{N}{2} \right\rceil}{2\left\lceil \frac{N}{2} \right\rceil - 1}$$

Idea: Repeat words of $\mathcal{W}_N$ periodicaly

Proposition 6 *There is an infinite set $\mathcal{X}_\infty$ of periodic configurations such that*

$$\forall x \in \mathcal{X}_\infty, \forall y \in \mathcal{X}_\infty, \mathbf{d}_B(x,y) = \frac{1}{2}$$